STABLE OPERATIONS IN MOTIVIC HOMOTOPY THEORY

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1. Overview

In this short note we will show how to compare two kinds of power operations in motivic cohomology. This has been temporarily extracted from forthcoming and long overdue work on the cohomology of motivic Eilenberg–MacLane spaces. More details and context will eventually be included there.

1.1. **Spectra of operations.** Let S be a scheme and $A \in \mathrm{NAlg}(\mathcal{SH}(S))$ a normed spectrum in the sense of [BH21]. Additionally assume that A is oriented (i.e., has Thom isomorphisms). Let $D : \mathcal{SH}(S) \to \mathcal{SH}(S)$ be an endofunctor, with diagonal transformations for suspension spectra as in [BEH22, Lemma 3.10] and Thom isomorphisms as in [BEH22, Proposition 5.37]. In fact the only cases we have in mind are $A = H\mathbb{F}_p$ the motivic cohomology spectra, and $D = D_\ell^{\mathrm{mot}}$ or D_ℓ^{naive} either the motivic or the categorical extended power functors for the group Σ_ℓ (see [BEH21, §5] for the former). As in [BEH22, §4.4] we have a functor $C : \mathbb{Z} \times \mathbb{Z} \to \mathcal{SH}(S)$ with value

$$C(m,n) = S^{-m} \otimes \mathbb{G}_m^{\wedge -n} \otimes D(S^m \otimes \mathbb{G}_m^{\wedge n}),$$

and transition maps coming from the diagonal maps. From this we can extract spectra of operations, as follows.

Definition 1.1. We set

$$\mathrm{Ops}^{\mathbb{P}^1}(D,A) = \lim_{\mathbb{Z}\times\mathbb{Z}}(C \otimes A)$$

and

$$\mathrm{Ops}^{S^1}(D,A) = \lim_{\mathbb{Z}\times 0}(C \otimes A).$$

This construction is clearly functorial in D and A, and moreover there is an evident map from $\mathrm{Ops}^{\mathbb{P}^1}(D,A)$ to $\mathrm{Ops}^{S^1}(D,A)$. Much of this note will be concerned with contemplating (the effect on homotopy) of the span

$$\mathrm{Ops}^{\mathbb{P}^1}(D_\ell^{\mathrm{mot}}, H\mathbb{F}_\ell) \to \mathrm{Ops}^{S^1}(D_\ell^{\mathrm{mot}}, H\mathbb{F}_\ell) \leftarrow \mathrm{Ops}^{S^1}(D_\ell^{\mathrm{naive}}, H\mathbb{F}_\ell).$$

1.2. **Construction of operations.** Fix a map $a : S^{m,n} \to \mathrm{Ops}^{S^1}(D,A)$. We shall explain how to use this to extract an operation in the A-cohomology of S^1-spectra.

Let $X \in \mathcal{S}\mathrm{pc}(S)_*$ and $x : \Sigma^p\Sigma^\infty X \to \Sigma^{2q,q}A$. We denote by $Q^a(x)$ the composite

$$\Sigma^p\Sigma^\infty X \otimes S^{m,n} \xrightarrow{\mathrm{id}\otimes a} \Sigma^p\Sigma^\infty X \otimes \mathrm{Ops}^{S^1}(D,A) \xrightarrow{(1)} \Sigma^p\Sigma^\infty X \otimes \Sigma^{-p}D(\Sigma^p\mathbb{1}) \otimes A$$
$$\xrightarrow{(2)} D(\Sigma^p\Sigma^\infty X) \otimes A \xrightarrow{D(x)} D(\Sigma^{2q,q}A) \otimes A \xrightarrow{(3)} \Sigma^{2\ell q,\ell q}A,$$

where the maps are explained as follows:

(1) projection from the limit to the appropriate term in the diagram defining $\mathrm{Ops}^{S^1}(D,A)$
(2) cancel off $\Sigma^p\Sigma^{-p}$ and diagonal for X into D
(3) Thom isomorphism $D(\Sigma^{2q,q}A) \simeq \Sigma^{2\ell q,\ell q}D(A)$ composed with the symmetric multiplication $D(A) \to A$ and the usual multiplication $A \otimes A \to A$.

This is natural in X. Moreover, by design this construction is compatible with suspension isomorphisms: if $x' : \Sigma^{p-1}\Sigma^\infty(\Sigma X) \to \Sigma^{2q,q}A$ is the map corresponding to x, then $Q^a(x')$ corresponds to $Q^a(x)$. Now let $E = (E^0, E^1, \dots) \in \mathcal{SH}^{S^1}(S)$, so that $E^i \in \mathcal{S}\mathrm{pc}(S)_*$ and we have bonding maps $\Sigma E^i \to E^{i+1}$. A map $x : \Sigma^p\sigma^\infty(E) \to \Sigma^{2q,q}A$ consists of compatible maps $x^i : \Sigma^{p-i}\Sigma^\infty(E^i) \to \Sigma^{2q,q}$. Applying Q^a, we obtain compatible maps $Q^a(x^i) : \Sigma^{p-i}\Sigma^\infty X \otimes S^{m,n} \to \Sigma^{2\ell q,\ell q}A$ and thus all in all a map

$$Q^a(x) : \Sigma^{p-i}\sigma^\infty(E) \otimes S^{m,n} \to \Sigma^{2\ell q,\ell q}A.$$

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Taking $p = 0$, $E = \omega^\infty \Sigma^{2q,q} A$ and x the unit of adjunction, we obtain by adjunctions a map

$$Q^a : \omega^\infty \Sigma^{2q,q} A \to \omega^\infty \Sigma^{2\ell q - m, \ell q - n} A.$$

This induces all the previous maps, by naturality.

Remark 1.2. In fact, for each n this construction induces

$$Q : \mathrm{map}_{\mathcal{SH}(S)}(\Sigma^{0,n}\mathbb{1}, \mathrm{Ops}^{S^1}(D, A)) \to \mathrm{map}_{\mathcal{SH}^{S^1}(S)}(\omega^\infty \Sigma^{2q,q} A, \omega^\infty \Sigma^{2\ell q, \ell q - n} A).$$

Then Q^a is the effect of Q applied to $a \in \pi_m \mathrm{map}_{\mathcal{SH}(S)}(\Sigma^{0,n}\mathbb{1}, \mathrm{Ops}^{S^1}(D, A))$.

Remark 1.3. We see that for the normed spectrum A (with Thom isomorphism), $\mathrm{Ops}^{S^1}(D_\ell^{\mathrm{mot}}, A)$ parametrizes (some) S^1-stable cohomology operations for A. When using $\mathrm{Ops}^{S^1}(D_\ell^{\mathrm{naive}}, A)$ instead, we in fact only need A to be an $\mathcal{E}_\infty$-ring, and the construction recovers the usual (topological) operations.

Alternatively, a similar construction can be performed with $\mathrm{Ops}^{\mathbb{P}^1}(D, A)$ instead. Then one in fact obtains $\mathbb{P}^1$-stable operations and in the end a map

$$Q : \mathrm{Ops}^{\mathbb{P}^1}(D, A) \to \underline{\mathrm{map}}_{\mathcal{SH}(S)}(A, A).$$

1.3. **Relating the operations.** Let us assume in addition that the functor D is oplax symmetric monoidal, as is the case in all our examples [BEH21, Proposition 6.12].

Construction 1.4. We get a coaction map $D(X) \to D(X) \otimes D(\mathbb{1})$ for every $X \in \mathcal{SH}(S)$, and these maps assemble to a coaction

$$\mathrm{Ops}^{S^1}(D, A) \to \mathrm{Ops}^{S^1}(D, A) \otimes_A (D(\mathbb{1}) \otimes A).$$

Dualizing (carefully), we see that $\pi_{**}\mathrm{Ops}^{S^1}(D, A)$ is a module over $\pi_{**}\underline{\mathrm{map}}(D(\mathbb{1}), A) \simeq A^{-*,-*}(D(\mathbb{1}))$. Similarly for $\mathrm{Ops}^{\mathbb{P}^1}(D, A)$.

It will thus be fruitful to understand the cohomology of $D_\ell^{\mathrm{naive}}(\mathbb{1})$ and $D_\ell^{\mathrm{mot}}(\mathbb{1})$.

Lemma 1.5. *Suppose $1/\ell \in S$.*

(1) Both $H\mathbb{F}_\ell \otimes D_\ell^{\mathrm{naive}}(\mathbb{1})$ and $H\mathbb{F}_\ell \otimes D_\ell^{\mathrm{mot}}(\mathbb{1})$ are split Tate.

(2) We have

$$H^{**}(D_\ell^{\mathrm{naive}}(\mathbb{1}), \mathbb{F}_\ell) \simeq H^{**}(S)[\![c_0, d_0]\!]/c_0^2 = R_0(d_0),$$

where $|c_0| = (2\ell - 3, 0)$, $|d_0| = (2\ell - 2, 0)$, $R_0(d_0) = d_0$ if $\ell = 2$ and $R_0(d_0) = 0$ else.

(3) The canonical map $H^{**}(D_\ell^{\mathrm{mot}}(\mathbb{1}), \mathbb{F}_\ell) \to H^{**}(D_\ell^{\mathrm{naive}}(\mathbb{1}), \mathbb{F}_\ell)$ is injective, onto the subring generated by

$$c = \tau^{\ell-1} c_0 \text{ and } d = \tau^{\ell-1} d_0 + \rho c_0.$$

Here $\rho = [-1]$ (which is zero mod ℓ if $\ell \neq 2$) and $\tau^{\ell-1} \in H^{0,\ell-1}(S, \mathbb{F}_p) \simeq H^0_{\acute{e}t}(S, \mu_\ell^{\otimes \ell-1})$ denotes the canonical generator.

Now we can state our main result.

Theorem 1.6. *Suppose $1/\ell \in S$.*

(1) Each of $\mathrm{Ops}^{S^1}(D_\ell^{\mathrm{naive}}, H\mathbb{F}_\ell)$, $\mathrm{Ops}^{S^1}(D_\ell^{\mathrm{mot}}, \mathbb{F}_\ell)$ and $\mathrm{Ops}^{\mathbb{P}^1}(D_\ell^{\mathrm{mot}}, \mathbb{F}_\ell)$ is a split Tate $H\mathbb{F}_\ell$-module.

(2) $\pi_{**}\mathrm{Ops}^{S^1}(D_\ell^{\mathrm{naive}}, H\mathbb{F}_\ell)$ has a basis $\{a_i^t, b_i^t\}_{i \in \mathbb{Z}}$ with $|a_i^t| = (i(2\ell-2), 0)$ and $|b_i^t| = (i(2\ell-2) - 1, 0)$. We have $d_0 a_{i+1}^t = a_i^t$ and $c_0 b_{i+1}^t = a_i^t$.

(3) $\pi_{**}\mathrm{Ops}^{\mathbb{P}^1}(D_\ell^{\mathrm{mot}}, H\mathbb{F}_\ell)$ has a basis $\{a_i^v, b_i^v\}_{i \in \mathbb{Z}}$ with $|a_i^v| = (i(2\ell-2), i(\ell-1))$ and $|b_i^v| = (i(2\ell-2) - 1, i(\ell-1))$. We have $da_{i+1}^v = a_i^v$ and $cb_{i+1}^v = a_i^v$.

(4) $\pi_{**}\mathrm{Ops}^{S^1}(D_\ell^{\mathrm{mot}}, H\mathbb{F}_\ell)$ has basis the images of a_i^t for $i < 0$, b_i^t for $i \leq 0$, a_i^v for $i \geq 0$ and b_i^v for $i > 0$. The elements a_0^t and a_0^v have the same image. The action of $c^\epsilon d^n$ is injective (for $\epsilon \in \{0,1\}, d \in \mathbb{Z}$).

We defer the proofs of these results to the next section.

Example 1.7. Suppose that $\ell \neq 2$ or -1 is a square in S, so that $\rho = 0$ in $H\mathbb{F}_2$.

(1) In $\pi_{**}\mathrm{Ops}^{\mathbb{P}^1}(D_\ell^{\mathrm{mot}}, H\mathbb{F}_\ell)$ we have $a_{-i}^v = d^i a_0$, for $i \geq 0$. In comparison in $\pi_{**}\mathrm{Ops}^{S^1}(D_\ell^{\mathrm{naive}}, H\mathbb{F}_\ell)$ we have

$$d^i a_0 = (\tau^{\ell-1} d_0)^i a_0 = \tau^{i(\ell-1)} a_{-i}^t.$$

Consequently in $\pi_{**}\mathrm{Ops}^{S^1}(D_\ell^{\mathrm{mot}}, H\mathbb{F}_\ell)$ we have

$$a_{-i}^v = \tau^{i(\ell-1)} a_{-i}^t, i \geq 0.$$

(2) Now let $i \geq 0$. We compute
$$d^i \tau^{i(\ell-1)} a_i^v = \tau^{i(\ell-1)} a_0 = \tau^{i(\ell-1)} d_0^i a_i^t = d^i a_i^t.$$
Since the action of d^i is injective, we deduce that
$$a_i^t = \tau^{i(\ell-1)} a_i^v, i > 0.$$
(3) The b_i are treated similarly.

Remark 1.8. Fix q. By construction the operation
$$Q^{a_i^v} : H^{*,q}(X, \mathbb{F}_\ell) \to H^{*+2(\ell-1)(q-i), q+(\ell-1)(q-i)}(X, \mathbb{F}_\ell)$$
is Voevodsky's B_v^{q-i}. Similarly, the operation
$$Q^{a_i^t} : H^{*,q}(X, \mathbb{F}_\ell) \to H^{*+2(\ell-1)(q-i), 2q}$$
is the topological operation B_t^{q-i}.

Corollary 1.9. *Suppose that $\ell \neq 2$ or -1 is a square in S. As S^1-stable operations acting on weight q cohomology, for $0 \leq i \leq q$ we have*
$$B_t^i = \tau^{(q-i)(\ell-1)} B_v^i.$$
On the other hand for $i \geq q$ we have
$$\tau^{(i-q)(\ell-1)} B_t^i = B_v^i.$$
The same holds for the βB^i.

Proof. Immediate from Example 1.7 and Remark 1.8. □

2. Proofs

Proposition 2.1. *Let S be any scheme. There are canonical classes*
$$u \in H^{1,1}(B_{\mathrm{fppf}} \mu_\ell, \mathbb{F}_\ell)$$
and
$$v \in H^{2,1}(B_{\mathrm{fppf}} \mu_\ell, \mathbb{F}_\ell)$$
such that
$$H^{**}(B_{\mathrm{fppf}} \mu_\ell, \mathbb{F}_\ell) \simeq H^{**}(S)[\![u, v]\!]/u^2 = R(u, v).$$
If $\ell = 2$ we have $R(u, v) = \tau v + \rho u$ and else we have $R = 0$. Moreover $B_{\mathrm{fppf}} \mu_\ell \wedge H\mathbb{F}_\ell$ is split Tate and $\partial(u) = v$.

Proof. The additive structure is [Spi18, Theorem 10.16]. It remains to determine u^2. The case $\ell = 2$ is [BEH22, Lemma 4.21]. If ℓ is odd then $u^2 = 0$ for degree reasons (i.e., given $u, u' \in H^{2,1}$ one has $uu' = -u'u$, so $2u^2 = 0$). □

From now on ℓ is a fixed prime invertible on S.

Proposition 2.2. *There are canonical classes*
$$c \in H^{2\ell-3, \ell-1}(D_\ell^{\mathrm{mot}}(S^0), \mathbb{F}_\ell)$$
and
$$d \in H^{2\ell-2, \ell-1}(D_\ell^{\mathrm{mot}}(S^0), \mathbb{F}_\ell)$$
such that
$$H^{**}(D_\ell^{\mathrm{mot}}(S^0), \mathbb{F}_\ell) \simeq H^{**}(S)[\![c, d]\!]/c^2 = R(c, d).$$
If $\ell = 2$ we have $R(c, d) = \tau d + \rho c$ and else we have $R = 0$. Moreover $D_\ell^{\mathrm{mot}}(S^0) \wedge H\mathbb{F}_\ell$ is split Tate and $\partial(c) = d$.

Proof. The case $\ell = 2$ is handled in Proposition 2.1, so we may assume ℓ odd. By stability of extended powers under base change, we may assume that $S = \mathrm{Spec}(\mathbb{Z}[1/\ell])$. We take d to be the Euler class of the tautological (permutation) representation. Note that in fact d is the reduction mod ℓ of an integrally defined class $\tilde{d}$.

We first argue that $\ell \tilde{d} = 0$ in cohomology with $\mathbb{Z}_{(\ell)}$ coefficients. By a transfer argument, for this we may replace S by $\mathrm{Spec}(\mathbb{Z}[1/\ell, \mu_\ell])$. Over this base we have $\mu_\ell \simeq \mathbb{Z}/\ell$. Since $\mathbb{Z}/\ell \subset S_\ell$ is a Sylow ℓ-subgroup, the map
$$H^{**}(B_{\acute{e}t} S_\ell, \mathbb{Z}_{(\ell)}) \to H^{**}(B_{\acute{e}t} \mu_\ell, \mathbb{Z}_{(\ell)})$$
is an injection, whence it suffices to prove the claim in the latter group. As in [Voe03, Lemma 6.13], the image of $\tilde{d}$ is a unit multiple of $\tilde{v}^{\ell-1}$, where $\tilde{v}$ is the integral Euler class of the tautological representation

of μ_ℓ. Finally note that $\ell\tilde{v}$ is the Euler class of the ℓ-th tensor power of the tautological representation of μ_ℓ, which is trivial as needed.

Having established the claim, we go back to the case $S = \mathrm{Spec}(\mathbb{Z}[1/\ell])$. It follows that there exists a class $c' \in H^{**}(B_{\acute{e}t}S_\ell, \mathbb{F}_\ell)$ with Bockstein $\tilde{d}$. Since $H^{**}(B_{\acute{e}t}S_\ell, \mathbb{F}_\ell) \to H^{**}(*, \mathbb{F}_\ell)$ is split, we can choose c' in such a way that its restriction to the base point vanishes; denote the resulting class by c. As before we have $c^2 = 0$ for degree reasons. Now we will show that the canonical map

$$B_{\acute{e}t}S_\ell \wedge H\mathbb{F}_\ell \to H\mathbb{F}_\ell\{c^\epsilon d^i \mid \epsilon \in \{0,1\}, i \in \mathbb{Z}_{\ge 0}\}$$

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is an equivalence. It suffices to check this after base change to the residue fields, which reduces to [Voe03, Theorem 6.16].

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The only remaining point is to argue why c is canonical. We claim that in fact c is the *unique* class such that $c|_* = 0$ and $\partial(c) = d$. Indeed since $H^{p,q}(S, \mathbb{F}_\ell) = 0$ for $p < 0$ we find that

$$H^{2\ell-3,\ell-1}(D_\ell^{\mathrm{mot}}(S^0), \mathbb{F}_\ell) \simeq H^{2\ell-3,\ell-1}(*, \mathbb{F}_\ell) \oplus \mathbb{F}_\ell\{c\},$$

which implies the claim. □

Proof of Lemma 1.5. The canonical functor $D(\mathbb{F}_\ell) \to H\mathbb{F}_\ell\text{-}\mathcal{M}\mathrm{od}$ sends $B\Sigma_\ell$ to $D_\ell^{\mathrm{naive}}(S^0) \wedge H\mathbb{F}_\ell$. Thus (1) and (2) hold. We must prove (3), i.e., determine the map; for this we may work over $\mathrm{Spec}(\mathbb{Z}[1/\ell])$. Since this has Zariski cohomological dimension 1, we have $H^{p,q}(*, \mathbb{F}_\ell) = 0$ for $p > q+1$. Also by construction, $H^{0,i(\ell-1)}(*, \mathbb{F}_\ell) = \mathbb{F}_\ell\{\tau^{i(\ell-1)}\}$. Thus for $\ell \ne 2$ we must have $c \mapsto a\tau^{\ell-1}c_0$ for some $a \in \mathbb{F}_\ell$.

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Considering complex realization we find $a = 1$, as needed. Similarly for d.

For $\ell = 2$, we see that c must map to $a\tau \cdot c_0 + b \cdot 1$. Restricting to the point we see that $b = 0$. Complex realization shows that $a = 1$. The claim for d_0 follows by applying the Bockstein. □

In order to deal with extended powers of other spheres, we will need the following.

Proposition 2.3. *Let $m, n \in \mathbb{Z}$ and write D_ℓ for either D_ℓ^{naive} or D_ℓ^{mot}.*

(1) If m is odd, then we have a cofiber sequence

$$\Sigma D_\ell(S^{m,n}) \otimes H\mathbb{F}_\ell \to D_\ell(S^{m+1,n}) \otimes H\mathbb{F}_\ell \to S^{\ell(m+1),\ell n} \otimes H\mathbb{F}_\ell.$$

(2) If m is even, then we have a cofiber sequence

$$\Sigma D_\ell(S^{m,n}) \otimes H\mathbb{F}_\ell \to D_\ell(S^{m+1,n}) \otimes H\mathbb{F}_\ell \to S^{2+\ell m,\ell n} \otimes H\mathbb{F}_\ell.$$

The proof is deferred to the next section. For now we show how to deduce the main result.

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Proof of Theorem 1.6. (3) We have a Thom isomorphism $D_\ell^{\mathrm{mot}}(\Sigma^{2,1}X) \otimes H\mathbb{F}_\ell \simeq \Sigma^{2\ell,\ell}D_\ell^{\mathrm{mot}}(X) \otimes H\mathbb{F}_\ell$. It follows that in the inverse system defining $\mathrm{Ops}^{\mathbb{P}^1}(D_\ell^{\mathrm{mot}}, H\mathbb{F}_\ell)$, we know the homology of all spectra via Lemma 1.5. Tracking the maps as in [BEH22, §4.2] establishes the claim.

(2) This reduces immediately to a statement in topology, which is proved in essentially the same way as (3).

(1) This follows from (2–4).

(4) We shall show that $D_\ell^{\mathrm{mot}}(S^{-s}) \wedge H\mathbb{F}_\ell$ is split Tate and that the map

$$H^{**}(D_\ell^{\mathrm{mot}}(S^{-s}, H\mathbb{F}_\ell) \to H^{**}(\mathrm{Ops}^{S^1}(D_\ell^{\mathrm{naive}}, H\mathbb{F}_\ell)$$

is this totally true for the action?

is an injection with the expected image. This implies what we need.

We work in $H\mathbb{F}_\ell$-modules throughout. We may assume $S = \mathrm{Spec}(\mathbb{Z}[1/\ell])$. The case $s = 0$ holds by Lemma 1.5.

First consider $s = -1$. Using Proposition 2.3, we obtain (by rotation) a diagram of cofiber sequences

$$\begin{tikzcd} S^0 \arrow[r] & D_\ell^{\mathrm{mot}}(S^0) \arrow[r] & \Sigma^{-1}D_\ell^{\mathrm{mot}}(S^1) \\ S^0 \arrow[u, equal] \arrow[r, "\partial"] & D_\ell^{\mathrm{naive}}(S^0) \arrow[u] \arrow[r] & \Sigma^{-1}D_\ell^{\mathrm{naive}}(S^1). \arrow[u] \end{tikzcd}$$

The map ∂ composed with the projection $D_\ell^{\mathrm{naive}}(S^0) \to S^0$ is fact an equivalence: it arises in topology (via the functor $D(\mathbb{F}_\ell) \to H\mathbb{F}_\ell\text{-}\mathcal{M}\mathrm{od}$), and the claim is easily verified by considering the long exact sequence and knowing $(S^1)^{\wedge \ell}_{h\Sigma_\ell}$. This implies that $\Sigma^{-1}D_\ell^{\mathrm{mot}}(S^1)$ is a summand of $D_\ell^{\mathrm{mot}}(S^0)$, its cohomology is the kernel of

$$H^{**}(D_\ell^{\mathrm{mot}}(S^0), \mathbb{F}_\ell) \to H^{**}(S^0, \mathbb{F}_\ell),$$

and is spanned by $\{c^\epsilon d^n\}$ except for $c^0 d^0$, as needed.

this is a bit of a mess

Now we treat $s > 0$ by induction. Again using Proposition 2.3 we obtain a diagram of cofiber sequences

$$\begin{tikzcd}
S^? \arrow[r] \arrow[d, equal] & \Sigma^{s+1} D_\ell^{\mathrm{mot}}(S^{-s-1}) \arrow[r] & \Sigma^s D_\ell^{\mathrm{mot}}(S^{-s}) \arrow[r, "\partial'"] & S^{?+1} \\
S^? \arrow[r] & \Sigma^{s+1} D_\ell^{\mathrm{naive}}(S^{-s-1}) \arrow[r] \arrow[u] & \Sigma^s D_\ell^{\mathrm{naive}}(S^{-s}) \arrow[r, "\partial"] \arrow[u] & S^{?+1}. \arrow[u, equal]
\end{tikzcd}$$

Since S is the spectrum of a Dedekind domain, and since by induction we know $D_\ell^{\mathrm{mot}}(S^{-s})$, we find (using [BEH22, Proposition D.2(3,4)]) that

$$[\Sigma^s D_\ell^{\mathrm{mot}}(S^{-s}), S^{?+1}] \simeq [\Sigma^s D_\ell^{\mathrm{naive}}(S^{-s}), S^{?+1}].$$

Again since ∂ arises from topology, it is easily verified to be 0. Thus $\partial' = 0$ and hence $\Sigma^{s+1} D_\ell^{\mathrm{mot}}(S^{-s-1})$ is split Tate. Arguing as before we find now that also

$$[\Sigma^{s+1} D_\ell^{\mathrm{mot}}(S^{-s-1}), S^?] \simeq [\Sigma^{s+1} D_\ell^{\mathrm{naive}}(S^{-s-1}), S^?],$$

from which we deduce that

$$H^{**}(\Sigma^{s+1} D_\ell^{\mathrm{mot}}(S^{-s-1}), \mathbb{F}_\ell) \to H^{**}(\Sigma^{s+1} D_\ell^{\mathrm{naive}}(S^{-s-1}), \mathbb{F}_\ell)$$

is surjective (even an isomorphism) in degree $(?, 0)$. The map is also injective (in all degrees) as is verified by using the (split) exact sequence (arising by taking cohomology in the diagram of cofiber sequences) and the inductive hypothesis. In particular we have proved that

$$H^{**}(\Sigma^{s+1} D_\ell^{\mathrm{mot}}(S^{-s-1}), \mathbb{F}_\ell) \to H^{**}(\Sigma^{\infty} D_\ell^{\mathrm{naive}}(\Sigma^{-\infty}\mathbb{1}), \mathbb{F}_\ell)$$

is injective and its image in degree $(0, ?)$ coincides with the image of $H^{**}(\Sigma^{s+1} D_\ell^{\mathrm{naive}}(S^{-s-1}), \mathbb{F}_\ell)$. Using that it also contains the image of $H^{**}(\Sigma^s D_\ell^{\mathrm{mot}}(S^{-s}), \mathbb{F}_\ell)$ (which we have already determined), it is thus at least as big as needed. Let $x \in H^{?,0}(\Sigma^\infty D_\ell^{\mathrm{naive}}(\Sigma^{-\infty}\mathbb{1}), \mathbb{F}_\ell)$ denote the "newly arisen" monomial. Then we see from the (split) exact sequence that $H^{**}(\Sigma^{s+1} D_\ell^{\mathrm{mot}}(S^{-s-1}), \mathbb{F}_\ell)$ is generated as an H^{**}-module by x and the image of $H^{**}(\Sigma^s D_\ell^{\mathrm{mot}}(S^{-s}), \mathbb{F}_\ell)$. This proves what we need. $\square$

3. Proof of Proposition 2.3

3.1. **Equivariant homotopy types of regular representation spheres.** For a finite group G,

$$\mathcal{S}\mathrm{pc}_*^G := \mathcal{P}_\Sigma(\mathrm{Fin}_G)_*$$

is the category of pointed G-spaces. The orbits functor $\mathrm{Fin}_G \to \mathrm{Fin}, X \mapsto X/G$ induces by left Kan extension the *genuine orbits* functor

$$\mathcal{S}\mathrm{pc}_*^G \to \mathcal{S}\mathrm{pc}_*, X \mapsto X/G.$$

It preserves colimits and is symmetric monoidal. On appropriate models, it is just given by the ordinary quotient topological space. details?

We are interested in the case $G = \Sigma_n$. Given a partition

$$\{1, \ldots, n\} = A_1 \amalg \cdots \amalg A_r$$

we obtain a corresponding subgroup

$$\Sigma_{|A_1|} \times \cdots \times \Sigma_{|A_r|} \subset \Sigma_n.$$

Subgroups of this form are called *partition subgroups.* We denote by $S^{\underline{n}} \in \mathcal{S}\mathrm{pc}_*^{\Sigma_n}$ the sphere S^n with Σ_n acting by permuting factors, or in other words the regular representation sphere. Recall that we have a diagonal map $S^1 \to S^{\underline{n}}$.

Proposition 3.1. *Consider the cofiber*

$$S^1 \to S^{\underline{n}} \to C.$$

The following hold.

1. *C lies in the subcategory generated under colimits by the orbits of proper partition subgroups.*
2. *$C/\Sigma_n \simeq S^2$*
3. *There is a cofiber sequence $C' \to C \to \Sigma_{n+} \wedge S^n$, with C' in the subcategory generated by orbits of* non-trivial *proper partition subgroups.*

Proof. The functor $\mathcal{S}\mathrm{pc}_* \to \mathcal{S}\mathrm{pc}_*^{\Sigma_n}$, $X \mapsto X^{\wedge \underline{n}}$ preserves sifted colimits. We can write $S^1 = * \amalg_{S^0} *$ as a sifted colimit $S^1 \simeq |X_\bullet|$, where $X_i = (S^0)^{\vee i}$. Hence $S^{\underline{n}} \simeq |X_\bullet^{\wedge \underline{n}}|$, and the natural map $S^1 \to S^{\underline{n}}$ arises from the levelwise diagonal $X_i \to X_i^{\wedge \underline{n}}$. Write C_i for its cofiber. We have $X_i^{\wedge \underline{n}} \simeq \{1, \ldots, i\}_+^{\underline{n}}$. Stabilizers of points therein are given by partition subgroups, and the only points with full stabilizers are those where all coordinates are equal, i.e., the diagonal subset. This proves (1).

ref. see e.g. https://ncatlab.org/nlab/show/symmetric+product+of+circles

(2) Essentially by definition, $S^{\underline{n}}/\Sigma_n$ is the n-th *reduced* symmetric power of S^1. For all $i \geq 1$, the *unreduced* symmetric power of S^1 is equivalent to S^1, and hence $S^{\underline{n}}/\Sigma_n = *$. The result follows since $(-)/\Sigma_n$ preserves colimits and $S^1/\Sigma_n \simeq S^1$, the action being trivial.

(3) We have a natural transformation

$$F_1 \to \mathrm{id} : \mathrm{Fin}_G \to \mathrm{Fin}_G,$$

where $F_1(X) \subset X$ is the set of points with non-trivial stabilizers. Form the cofiber sequence

$$F_1' := \mathcal{P}_\Sigma(F_1)_* \to \mathrm{id} \to F_2 : \mathcal{S}\mathrm{pc}_*^G \to \mathcal{S}\mathrm{pc}_*^G.$$

Then F_2 preserves colimits (since F_1 preserves finite coproducts and $*$) and satisfies $F_2(G/H_+) = *$ for H non-trivial, $F_2(G_+) = G_+$.

Now revert to $G = \Sigma_n$ and set $C' = F_1'(C)$ and $C'' = F_2(C)$. Then C' lies in the appropriate subcategory by construction and (1), and it remains to determine C''. Observe first that we have $F_2(S^1) = *$ and hence $C'' \simeq F_2(S^{\underline{n}})$. Note next that we have the cofiber sequence

$$S(\bar\rho)_+ \wedge S^1 \to S^1 \to S^{\underline{n}},$$

whence

$$C \simeq \Sigma^2 S(\bar\rho)_+.$$

ref

We can identify $S(\bar\rho)$ with $\partial\Delta^{n-1} \subset \mathbb{R}^n$. Moreover, if we consider the barycentric subdivision $\mathcal{S}$ of $\partial\Delta^{n-1}$, then the Σ_n-action is even simplicial. The simplices of $\mathcal{S}$ are indexed by chains of proper subsets of $\{1, \dots, n\}$. If such a chain is not maximal then it is fixed by a transposition, conversely if the chain is maximal then it is not fixed by any element of Σ_n. We deduce that $F_2(S(\bar\rho)_+) \simeq F_2(S(\bar\rho))$ is obtained from $S(\bar\rho) \simeq \mathcal{S}$ by collapsing all simplices corresponding to non-maximal chains. This means we are left with a wedge of $(n-2)$-spheres, indexed by the maximal chains. A maximal chain is the same as an ordering of $\{1, \dots, n\}$, and hence we find that $F_2(S(\bar\rho)) \simeq S^{n-2} \wedge \Sigma_{n+}$, as needed. □

3.2. **Motivic extended powers.** We begin by recalling some terminology and constructions of [BEH21]. Given a presentably normed category over a scheme S, i.e.

$$\mathcal{C} \in \mathrm{Fun}^\times(\mathrm{Span}(\mathrm{Sm}_S, \mathrm{f\acute{e}t}, \mathrm{all})^{\mathrm{op}}, \mathcal{C}\mathrm{at}),$$

and $E \in \mathcal{C}(S)$ we obtain the *fundamental diagram*

$$E^{\underline{\otimes} n} : B_{\acute{e}t}\Sigma_n \to \mathcal{C}(-), (p : X \to Y) \mapsto p_\otimes(E|_X) \in \mathcal{C}(Y).$$

We also have the motivic colimit functor

$$\mathcal{P}(\mathrm{Sm}_S)_{/\mathcal{C}(S)^\simeq} \to \mathcal{C}(S).$$

Next consider the composite

$$Orb_{\Sigma_n} \subset \mathrm{Fin}_{\Sigma_n} \to \mathrm{Fin}_{/B\Sigma_n} \to \mathrm{Shv}_{\acute{e}t}(\mathrm{Sm}_S)_{/B_{\acute{e}t}\Sigma_n} \to \mathcal{P}(\mathrm{Sm}_S)_{/B_{\acute{e}t}\Sigma_n}.$$

Composing with $E^{\underline{\otimes} n}$ and M we obtain a functor

$$Orb_{\Sigma_n} \to \mathcal{C}(S).$$

Since the target is pointed and has colimits, we may extend this to

$$D_n^E : \mathcal{S}\mathrm{pc}_*^{\Sigma_n} \simeq \mathcal{P}(Orb_G)_* \to \mathcal{C}(S).$$

Example 3.2. By construction, we have

$$D_n^E(\Sigma_n/H_+) \simeq M(B_{\acute{e}t}H \to B_{\acute{e}t}\Sigma_n \xrightarrow{E^{\underline{\otimes} n}} \mathcal{C}(-)).$$

In particular $D_n^E(S^0) \simeq D_n(E)$ is the n-th motivic extended power of E.

Lemma 3.3. *We have $D_n^E(S^1) \simeq \Sigma D_n(E)$ and $D_n^E(S^{\underline{n}}) \simeq D_n(\Sigma E)$.*

Proof. The first statement follows from the second part of Example 3.2, since by construction D_n^E preserves colimits and zero objects. For the second statement, writing S^1 as a sifted colimit of finite pointed sets, it will suffice to establish an equivalence of the two functors

$$\mathrm{Fin}_+ \to \mathcal{SH}(S), \quad X \mapsto D_n(\Sigma^\infty X \otimes E) \quad \text{respectively} \quad X \mapsto D_n^E(X^{\wedge \underline{n}}).$$

rest of the argument is a bit sloppy

Using Lemma 3.4 below we find

$$D_n(\Sigma^\infty X \otimes E) \simeq \bigoplus_{a_1x_1 + \cdots + a_rx_r \in Sym^n(X)} D_{a_1}(E) \otimes \dots D_{a_r}(E).$$

On the other hand

$$X^{\wedge \underline{n}} \simeq \coprod_{a_1x_1+\cdots+a_rx_r \in Sym^n(X)} \Sigma_n/(\Sigma_{a_1} \times \cdots \times \Sigma_{a_r}).$$

We conclude by the first part of Example 3.2, together with the fact that the composite

$$B_{\acute{e}t}\Sigma_{a_1} \times \cdots \times B_{\acute{e}t}\Sigma_{a_r} \to B_{\acute{e}t}\Sigma_n \xrightarrow{E^{\otimes n}} \mathcal{C}(-))$$

has motivic colimit $D_{a_1}(E) \otimes \cdots \otimes D_{a_r}(E)$ (use [BEH21, Proposition 3.14]). □

We used the following relatively formal fact.

Lemma 3.4. *For X, Y in a presentably normed ∞-category $\mathcal{C}$, we have canonical equivalences*

$$D_n(X \oplus Y) \simeq \bigoplus_{i+j=n} D_i(X) \otimes D_j(Y).$$

Proof. This holds since $\mathrm{NSym} : \mathcal{C}(S) \to \mathrm{NAlg}(\mathcal{C}(S))$ preserves colimits, coproducts in $\mathrm{NAlg}(\mathcal{C}(S))$ are given by tensor products, and NSym is given by the sum of the D_n [Bac22, Theorem 3.10]. □

Definition 3.5. We call $E \in \mathcal{C}$ *even* (respectively *odd*) if the switch map on $E \otimes E$ is homotopic to id (respectively $-$id). We call E strongly even if in addition $\mathrm{Map}(E^{\otimes \ell}, E^{\otimes \ell})$ is 0-truncated.

Remark 3.6. Note that any sphere $\Sigma^{p,q} H\mathbb{F}_\ell$ is odd or even (as $H\mathbb{F}_\ell$-module). Moreover the relevant mapping space is 0-truncated at least if S is a Dedekind scheme (being given by $K_{\mathrm{Nis}}(\mathbb{F}_\ell, 0)$). ref

Proposition 3.7. *Let $\mathcal{C}(-) = H\mathbb{F}_\ell$-$\mathcal{M}$od for some prime ℓ.*

(1) Suppose that E is odd. Then in the notation of Proposition 3.1, the map $D_\ell^E(C) \to D_\ell^E(\Sigma_{n+} \wedge S^n)$ is an equivalence.
(2) Suppose that E is strongly even. Then on the subcategory of $\mathcal{S}\mathrm{pc}_*^{\Sigma_\ell}$ generated by orbits of proper partition subgroups, the functor D_ℓ^E is given by $(-)/\Sigma_\ell \otimes E^{\otimes \ell}$.

Proof. (1)We must prove that $D_\ell^E(X_+) = 0$ when X is the orbit of a non-trivial proper partition subgroup. For this it is enough to show that $D_i(E) = 0$ for $1 < i < \ell$, which follows from oddness by writing $D_i(E)$ as a summand of $E^{\otimes i}$. details

(2) Let $\mathcal{D} \subset Orb_{\Sigma_\ell}$ be the full subcategory on orbits of proper partition subgroups. Since both sides preserve colimits, it will be enough to establish an equivalence of functors $\mathcal{D} \to \mathcal{C}(S)$. Note that both functors take value constantly at $E^{\otimes \ell}$, which has 0-truncated mapping space by assumption. It thus suffices to establish the equivalence up to (non-coherent) homotopy, which is clear. □

Corollary 3.8. *Let $\mathcal{C}(-) = H\mathbb{F}_\ell$-$\mathcal{M}$od for some prime ℓ.*

(1) If E is odd, then we have a cofiber sequence

$$\Sigma D_\ell(E) \to D_\ell(\Sigma E) \to (\Sigma E)^{\otimes \ell}.$$

(2) If E is strongly even, then we have a cofiber sequence

$$\Sigma D_\ell(E) \to D_\ell(\Sigma E) \to \Sigma^2 E^{\otimes \ell}.$$

Proof. By Lemma 3.3 we have a cofiber sequence

$$\Sigma D_\ell(E) \to D_\ell(\Sigma E) \to D_\ell^E(C),$$

where C is the cofiber of $S^1 \to S^{\underline{\ell}}$. If E is odd then by Proposition 3.7(1) we get $D_\ell^E(C) \simeq \Sigma^\ell E^{\otimes \ell}$, as needed. If E is strongly even then by Proposition 3.7(2) we get $D_\ell^E(C) \simeq C/\Sigma_\ell \otimes E^{\otimes \ell}$, which has the desired form by Proposition 3.1. □

Proof of 2.3. We may assume $S = \mathrm{Spec}(\mathbb{Z}[1/\ell])$. The case $D_\ell = D_\ell^{\mathrm{mot}}$ is essentially Corollary 3.8, using Remark 3.6. The case $D_\ell = D_\ell^{\mathrm{naive}}$ is well-known (and can be established by an argument along the same lines). □

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