NOTES ON MOTIVIC INFINITE LOOP SPACE THEORY

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Abstract. In fall of 2019, the Thursday Seminar at Harvard University studied motivic infinite loop space theory. As part of this, the authors gave a series of talks outlining the main theorems of the theory, together with their proofs, in the case of infinite perfect fields. These are our extended notes on these talks.

Contents

1. The reconstruction theorem

Primary sources: [EHK$^+$19b, GP18a].

1.1. **Setup.** Recall [EHK$^+$19b, §4] that there is a symmetric monoidal ∞-category $\mathrm{Corr}^{\mathrm{fr}}(S)$ and a symmetric monoidal functor $\gamma : \mathrm{Sm}_{S+} \to \mathrm{Corr}^{\mathrm{fr}}(S)$.[1] It preserves finite coproducts and is essentially surjective. We denote by $\gamma^* : \mathcal{P}_\Sigma(\mathrm{Sm}_{S+}) \to \mathcal{P}_\Sigma(\mathrm{Corr}^{\mathrm{fr}}(S))$ its sifted cocontinuous extension.[2] Write $\mathcal{S}\mathrm{pc}(S)_*$ for the localization of $\mathcal{P}_\Sigma(\mathrm{Sm}_{S+})$ at the Nisnevich equivalences and the $\mathbb{A}^1$-homotopy equivalences, and $\mathcal{S}\mathrm{pc}^{\mathrm{fr}}(S)$ for the localization of $\mathcal{P}_\Sigma(\mathrm{Corr}^{\mathrm{fr}}(S))$ at the images of motivic equivalences under γ^*. Let $\mathbb{P}^1 \in \mathcal{S}\mathrm{pc}(S)_*$ be pointed at 1. Recall that for any presentably symmetric monoidal ∞-category $\mathcal{C}$ and any object $P \in \mathcal{C}$ there is a universal presentably symmetric monoidal ∞-category under $\mathcal{C}$ in which P becomes $\otimes$-invertible [Rob15, §2.1]; we denote it by $\mathcal{C}[P^{-1}]$.

The following is the main result of this section.

Date: December 5, 2019.

[1] Recall that for a category with finite coproducts and a final object $*$, $\mathcal{C}_+ \subset \mathcal{C}_{*/}$ denotes the subcategory on objects of the form $c \coprod *$. We mainly use this in conjunction with the equivalence $\mathcal{P}_\Sigma(\mathcal{C}_+) \simeq \mathcal{P}_\Sigma(\mathcal{C})_*$ [BH17, Lemma 2.1].

[2] We denote by $\mathcal{P}_\Sigma(\mathcal{C}) = \mathrm{Fun}^\times(\mathcal{C}^{\mathrm{op}}, \mathcal{S}\mathrm{pc})$ the non-abelian derived category of $\mathcal{C}$.

Theorem 1.1 (reconstruction)**.** *The induced functor*

$$\gamma^* : \mathcal{S}\mathrm{pc}(S)_*[(\mathbb{P}^1)^{-1}] \to \mathcal{S}\mathrm{pc}^{\mathrm{fr}}(S)[\gamma^*(\mathbb{P}^1)^{-1}]$$

is an equivalence.

We write $\mathcal{SH}(S) = \mathcal{S}\mathrm{pc}(S)_*[(\mathbb{P}^1)^{-1}]$ and $\mathcal{SH}^{\mathrm{fr}}(S) = \mathcal{S}\mathrm{pc}^{\mathrm{fr}}(S)[\gamma^*(\mathbb{P}^1)^{-1}]$. We shall prove the result when $S = Spec(k)$ is the spectrum of a perfect field. The result for general S is reduced to this case in [Hoy18].

1.2. **Preliminary reductions.** The functor γ^* preserves colimits by construction, so has a right adjoint γ_*. The stable presentable ∞-category $\mathcal{SH}(S)$ is compactly generated by objects of the form $\Sigma^\infty_+ X \wedge (\mathbb{P}^1)^{\wedge n}$, for $X \in \mathrm{Sm}_S$ and $n \in \mathbb{Z}$. Similarly $\mathcal{SH}^{\mathrm{fr}}(S)$ is compactly generated by $\gamma^*(\Sigma^\infty_+ X \wedge (\mathbb{P}^1)^{\wedge n})$. It follows that $\gamma_* : \mathcal{SH}^{\mathrm{fr}}(S) \to \mathcal{SH}(S)$ is conservative and preserves colimits.

Conservativity of γ_* implies that in order to prove that γ^* is an equivalence, it suffices to show that it is fully faithful, or equivalently that the unit of adjunction $u : \mathrm{id} \to \gamma_*\gamma^*$ is an equivalence. Indeed the composite

$$\gamma_* \xrightarrow{u\gamma_*} \gamma_*\gamma^*\gamma_* \xrightarrow{\gamma_* c} \gamma_*$$

is the identity (γ^* and γ_* being adjoints), the first transformation is an equivalence by assumption, hence so is the second one, and finally so is the counit c since γ_* is conservative.

Since γ_* preserves colimits, the class of objects on which u is an equivalence is closed under colimits. Hence it suffices to show that u is an equivalence on the generators.

Given any adjunction $F : \mathcal{C} \leftrightarrows \mathcal{D} : U$ with F symmetric monoidal, the right adjoint U satisfies a projection formula for strongly dualizable objects: if $P \in \mathcal{C}$ is strongly dualizable, then there is an equivalence of functors $\gamma_*(- \otimes \gamma^* P) \simeq \gamma_*(-) \otimes P$. Indeed we have a sequence of binatural equivalences

$$\begin{gathered}\mathrm{Map}(-, \gamma_*(- \otimes \gamma^* P)) \simeq \mathrm{Map}(\gamma^*(-), - \otimes \gamma^* P) \\ \simeq \mathrm{Map}(\gamma^*(- \otimes P^\vee), -) \simeq \mathrm{Map}(- \otimes P^\vee, \gamma_*(-)) \simeq \mathrm{Map}(-, \gamma_*(-) \otimes P),\end{gathered}$$

and hence the result follows by the Yoneda lemma.

Since $\Sigma^\infty \mathbb{P}^1 \in \mathcal{SH}(S)$ is invertible and hence strongly dualizable, in order to prove Theorem 1.1 it is thus enough to show that for every $X \in \mathrm{Sm}_S$, the unit map

$$\Sigma^\infty_+ X \to \gamma_*\gamma^*\Sigma^\infty_+ X \in \mathcal{SH}(S)$$

is an equivalence. Using Zariski descent, we may further assume that X is affine.

1.3. **Recollections on prespectra.** Let $\mathcal{C}$ be a presentably symmetric monoidal ∞-category, and $P \in \mathcal{C}$. We denote by $\mathcal{S}\mathrm{p}^{\mathbb{N}}(\mathcal{C}, P)$ the ∞-category whose objects are sequences $(X_1, X_2, \dots)$ with $X_i \in \mathcal{C}$, together with "bonding maps" $P \otimes X_i \to X_{i+1}$. The morphisms are the evident commutative diagrams. We call $X = (X_n)_n \in \mathcal{S}\mathrm{p}^{\mathbb{N}}(\mathcal{C}, P)$ an Ω-spectrum if the adjoints of the bonding maps, $X_i \to \Omega_P X_{i+1}$, are all equivalences. We denote by $L_{\mathrm{st}}\mathcal{S}\mathrm{p}^{\mathbb{N}}(\mathcal{C}, P) \subset \mathcal{S}\mathrm{p}^{\mathbb{N}}(\mathcal{C}, P)$ the subcategory of Ω-spectra. The inclusion has a left adjoint which we denote by L_{st}; the maps inverted by L_{st} are called stable equivalences.

Remark 1.2. If P is a symmetric object, i.e. for some $n \geq 2$ the cyclic permutation on $P^{\otimes n}$ is homotopic to the identity, then $L_{\mathrm{st}}\mathcal{S}\mathrm{p}^{\mathbb{N}}(\mathcal{C}, P) \simeq \mathcal{C}[P^{-1}]$. This is proved in [Rob15, Corollary 2.22].

1.3.1. *Spectrification.* There is a natural transformation

$$\Sigma_P \Omega_P \xrightarrow{c} \mathrm{id} \xrightarrow{u} \Omega_P \Sigma_P.$$

Using this we can build a functor $Q_1 : \mathcal{S}\mathrm{p}^{\mathbb{N}}(\mathcal{C}, P) \to \mathcal{S}\mathrm{p}^{\mathbb{N}}(\mathcal{C}, P)$ with the property that for $X = (X_n)_n \in \mathcal{S}\mathrm{p}^{\mathbb{N}}(\mathcal{C}, P)$ we have $Q_1(X)_n = \Omega_P X_{n+1}$. Moreover there is a natural transformation $\mathrm{id} \to Q_1$. Iterating this construction and taking the colimit we obtain

$$\mathrm{id} \to Q := \operatorname*{colim}_n Q_1^{\circ n}.$$

The following is well-known.

Lemma 1.3. *Let $X \in \mathcal{S}\mathrm{p}^{\mathbb{N}}(\mathcal{C}, P)$.*

(1) The map $X \to QX$ is a stable equivalence.
(2) If Ω_P preserves filtered colimits (i.e. $P \in \mathcal{C}$ is compact), then QX is an Ω-spectrum.

1.3.2. *Prolongation.* Let $F : \mathcal{C} \to \mathcal{C}$ be an endofunctor. Following Hovey [Hov01, Lemma 5.2], we call F *prolongable* if we are provided with a natural transformation $\tau : \Sigma_P F \to F\Sigma_P$. Equivalently, we should provide a natural transformation $F \to \Omega_P F \Sigma_P$. In any case, there is an obvious category of prolongable endofunctors (having objects the pairs (F, τ) as above). Any prolongable functor (F, τ) induces an endofunctor

$$F : Sp^{\mathbb{N}}(\mathcal{C}, P) \to Sp^{\mathbb{N}}(\mathcal{C}, P), (X_n)_n \mapsto (FX_n)_n.$$

The structure maps of FX are given by

$$\Sigma_P F(X_n) \xrightarrow{\tau_{X_n}} F(\Sigma_P X_n) \xrightarrow{Fs_n} F(X_{n+1}).$$

Example 1.4. The functor $F_n = \Omega_P^n \Sigma_P^n$ is prolongable by $\Omega_P^n u \Sigma_P^n : F_n \to \Omega_P F_n \Sigma_P$, where $u : \mathrm{id} \to \Omega_P \Sigma_P$ is the unit transformation. One checks easily that

$$F_n \Sigma^\infty X \simeq Q_1^{\circ n} \Sigma^\infty X.$$

The transformation $\Omega_P^n u \Sigma_P^n$ defines a morphism $F_n \to F_{n+1}$ of prolongable functors; let F_∞ be its colimit. Then one checks that

$$F_\infty \Sigma^\infty X \simeq Q\Sigma^\infty X.$$

Example 1.5. The functor $F = \Sigma_P$ can be prolonged a priori in (at least) two ways, via the canonical isomorphism $\tau_1 : \Sigma_P F = \Sigma_P \Sigma_P = F\Sigma_P$ and via the switch map $\tau_2 : \Sigma_{P\otimes P} \to \Sigma_{P \otimes P}$. Then $F_1 \simeq F_2$ as prolongable functors if and only if the switch map on $P \otimes P$ is the identity.

Example 1.6. Let $F : \mathcal{C} \to \mathcal{C}$ be a lax $\mathcal{C}$-module functor, so that in particular for each $A \in \mathcal{C}$ we are given a transformation $\Sigma_A F \to F\Sigma_A$. Specializing to $A = P$ we obtain a prolongable functor $\tilde{F}$, natural in the lax $\mathcal{C}$-module functor F. The functor F_n (from Example 1.4) is a lax $\mathcal{C}$-module functor, via

$$A \otimes \underline{\mathrm{Hom}}(P^{\otimes n}, P^{\otimes n} \otimes X) \to \underline{\mathrm{Hom}}(P^{\otimes n}, P^{\otimes n} \otimes A \otimes X), \text{“}(a \otimes f) \mapsto c_a \otimes f\text{”},$$

where c_a denotes the “constant map at a”.

Suppose that P is strongly dualizable. Then the prolongation of F_n can be written as

$$P^{\vee \otimes n} \otimes P^{\otimes n} \xrightarrow{u} P^{\vee\otimes n} \otimes P^{\otimes n} \otimes P^\vee \otimes P \xrightarrow{\sigma_{324}} P^{\vee \otimes n} \otimes P^\vee \otimes P \otimes P^{\otimes n} \simeq P^{\vee \otimes n+1} \otimes P^{\otimes n+1}.$$

On the other hand the prolongation of $\tilde{F}_n$ can be written as

$$P^{\vee \otimes n} \otimes P^{\otimes n} \xrightarrow{u} P^{\vee\otimes n} \otimes P^{\otimes n} \otimes P^\vee \otimes P \xrightarrow{\sigma_{123}} P^\vee \otimes P^{\vee \otimes n} \otimes P^{\otimes n} \otimes P \simeq P^{\vee \otimes n+1} \otimes P^{\otimes n+1}.$$

They are isomorphic if and only if the $(n+1)$-fold cyclic permutation acts trivially on $P^{\otimes n}$.

Example 1.7. Let $\mathrm{id} \xrightarrow{u} F_1 \xrightarrow{\rho} \mathrm{id}$ be a retraction of prolongable functors. Then the following square commutes, by assumption

$$\begin{array}{ccc} \Omega_P \Sigma_P & \xrightarrow{\Omega_P u \Sigma_P} & \Omega_P^2 \Sigma_P^2 \\ {\scriptstyle\rho}\downarrow & & \downarrow{\scriptstyle \Omega_P \rho \Sigma_P} \\ \mathrm{id} & \xrightarrow{u} & \Omega_P \Sigma_P. \end{array}$$

Hence $u\rho \simeq \Omega_P \rho u \Sigma_P \simeq \mathrm{id}_{\Omega_P \Sigma_P}$ and so u and ρ are inverse equivalences.

Remark 1.8. As we have seen above, $\tilde{F}_1$ is actually a more natural prolongation in some sense, and so it is more natural to have a retraction $\mathrm{id} \xrightarrow{u'} \tilde{F}_1 \xrightarrow{\rho} \mathrm{id}$. If P is 2-symmetric (i.e. the switch on $P^{\otimes 2}$ is the identity), then $F_1 \simeq \tilde{F}_1$ and $u' \simeq u$, under this equivalence. Hence u', ρ are inverse equivalences. This holds more generally if P is n-symmetric for any $n \geq 2$; this is the content of Voevodsky's cancellation theorem. See Theorem 3.7 in §3.

1.4. **Equationally framed correspondences.**

1.4.1. *Framed correspondences.* We have the lax Sm_{S+}-module functor

$$h^{\mathrm{fr}} : \mathrm{Sm}_{S+} \to \mathcal{P}_\Sigma(\mathrm{Sm}_S)_*, X_+ \mapsto \gamma_* \gamma^* X_+.$$

We extend this to a sifted cocontinuous functor

$$h^{\mathrm{fr}} : \mathcal{P}_\Sigma(\mathrm{Sm}_S)_* \simeq \mathcal{P}_\Sigma(\mathrm{Sm}_{S+}) \to \mathcal{P}_\Sigma(\mathrm{Sm}_S)_*.$$

Of course $\gamma_*\gamma^*$ is already sifted cocontinuous, so $h^{\mathrm{fr}} \simeq \gamma_*\gamma^*$ and this is just a notational change.

1.4.2. *Equationally framed correspondences.* There are explicitly defined lax Sm_{S+}-module functors [EHK+19b, §2.1]

$$h^{\mathrm{efr},n} : \mathrm{Sm}_{S+} \to \mathcal{P}_\Sigma(\mathrm{Sm}_S)_*$$

and natural transformations $\sigma : h^{\mathrm{efr},n} \to h^{\mathrm{efr},n+1}$. We denote by

$$h^{\mathrm{efr},n} : \mathcal{P}_\Sigma(\mathrm{Sm}_S)_* \to \mathcal{P}_\Sigma(\mathrm{Sm}_S)_*$$

the sifted cocontinuous extensions, and by

$$h^{\mathrm{efr}} : \mathcal{P}_\Sigma(\mathrm{Sm}_S)_* \to \mathcal{P}_\Sigma(\mathrm{Sm}_S)_*$$

the colimit along σ. We will elaborate on this in §2.1.1.

1.4.3. *Relative equationally framed correspondences.* Let $U \subset X \in \mathrm{Sm}_S$ be an open immersion. There are explicitly defined presheaves

$$h^{\mathrm{efr},n}(X, U) \in \mathcal{P}_\Sigma(\mathrm{Sm}_{S+});$$

they depend functorially on the pair (X, U) and are lax modules, in a way which we will not elaborate on. For us the most important case is where $X = X' \times \mathbb{A}^m$ and $U = X' \times \mathbb{A}^m \setminus X' \times \{0\}$; we put

$$h^{\mathrm{efr},n}(X', \mathcal{O}^n) = h^{\mathrm{efr},n}(X' \times \mathbb{A}^m, X' \times \mathbb{A}^m \setminus X' \times \{0\}.$$

These assemble into lax Sm_{S+}-module functors $\mathrm{Sm}_{S+} \to \mathcal{P}_\Sigma(\mathrm{Sm}_{S+})$. We will elaborate on this in §2.1.

1.5. **Comparison results.**

1.5.1. *Equationally framed versus tangentially framed.* There is a canonical transformation

$$h^{\mathrm{efr}} \to h^{\mathrm{fr}} \in \mathrm{Fun}(\mathrm{Sm}_{S+}, \mathcal{P}_\Sigma(\mathrm{Sm}_{S+}))$$

which is a motivic equivalence (objectwise) [EHK+19b, Corollaries 2.2.20 and 2.3.25]. Since motivic equivalences are stable under (sifted) colimits, the sifted cocontinuous extension of the natural transformation is still a motivic equivalence objectwise. The transformations are compatible with the lax module structures.

1.5.2. *The cone theorem.* There is a canonical transformation

$$h^{\mathrm{efr},n}(X/U) \to h^{\mathrm{efr},n}(X, U);$$

here the left hand side is obtained by sifted cocontinuous extension. This is a motivic equivalence for X affine, provided the base is an infinite perfect field. This is known as the cone theorem, and will be treated in §2.

The natural transformation

$$h^{\mathrm{efr},n}(X \times \mathbb{A}^m / X \times \mathbb{A}^m \setminus X \times 0) \to h^{\mathrm{efr},n}(X, \mathcal{O}^m)$$

can be promoted to a lax module transformation.

1.5.3. *Voevodsky's lemma.* There is a canonical equivalence of lax module functors

$$h^{\mathrm{efr},n}(X, \mathcal{O}^m) \to \Omega^n_{\mathbb{P}^1} L_{\mathrm{Nis}} \Sigma^{n+m}_T X_+.$$

This is known as Voevodsky's Lemma, see [EHK+19b, Appendix A] for a proof. The equivalence is compatible with the natural stabilization maps (increasing n) on both sides.

1.6. **Proof of reconstruction.** Write $\mathcal{S}\mathrm{hv}_{\mathrm{Nis}}(S) = L_{\mathrm{Nis}}\mathcal{P}_\Sigma(\mathrm{Sm}_S)$ and similarly $\mathcal{S}\mathrm{hv}^{\mathrm{fr}}_{\mathrm{Nis}}(S) = L_{\mathrm{Nis}}\mathcal{P}_\Sigma(\mathrm{Corr}^{\mathrm{fr}}(S))$.

Lemma 1.9. *The forgetful functor $\mathcal{S}\mathrm{hv}^{\mathrm{fr}}_{\mathrm{Nis}}(S) \to \mathcal{S}\mathrm{hv}_{\mathrm{Nis}}(S)$ preserves and detects motivic equivalences.*

Proof. Immediate from [EHK+19b, Proposition 3.2.14]. □

Since $\gamma^* : \mathcal{S}\mathrm{hv}_{\mathrm{Nis}}(S)_* \to \mathcal{S}\mathrm{hv}^{\mathrm{fr}}_{\mathrm{Nis}}(S)$ is symmetric monoidal, it induces a functor $\gamma^*_{\mathbb{N}}$ upon passage to prespectra. We obtain an adjunction

$$\gamma^*_{\mathbb{N}} : \mathcal{S}\mathrm{p}^{\mathbb{N}}(\mathcal{S}\mathrm{hv}_{\mathrm{Nis}}(S)_*, \mathbb{P}^1) \leftrightarrows \mathcal{S}\mathrm{p}^{\mathbb{N}}(\mathcal{S}\mathrm{hv}^{\mathrm{fr}}_{\mathrm{Nis}}(S), \gamma^*\mathbb{P}^1) : \gamma^{\mathbb{N}}_*;$$

the right adjoint $\gamma^{\mathbb{N}}_*$ is given by the formula $\gamma^{\mathbb{N}}_*(X)_n \simeq \gamma_*(X_n)$. We call a map $X \to Y \in \mathcal{S}\mathrm{p}^{\mathbb{N}}(\mathcal{S}\mathrm{hv}_{\mathrm{Nis}}(S)_*, \mathbb{P}^1)$ a *level motivic equivalence* if each map $X_n \to Y_n$ is a motivic equivalence, and similarly for framed prespectra. The saturated class generated by level motivic equivalences and stable equivalences is called stable motivic equivalences. Local objects for this class of maps are called motivic Ω-spectra; these are the prespectra $X = (X_n)_n$ such that X is an Ω-spectrum and each X_n is motivically local.

Corollary 1.10. *The functor $\gamma^{\mathbb{N}}_*$ preserves and detects stable motivic equivalences.*

Proof. Since $\gamma_*^{\mathbb{N}}$ preserves motivic Ω-spectra it is enough to show that it commutes with spectrification. Let $X = (X_n)_n$ be a prespectrum. By Lemma 1.3(2), its spectrification is given by

$$(QL_{\mathrm{mot}}X)_n = \operatorname*{colim}_i \Omega^i_{\mathbb{P}^1} L_{\mathrm{mot}} X_{n+i}.$$

Since $\gamma_* : \mathcal{S}\mathrm{hv}^{\mathrm{fr}}_{\mathrm{Nis}}(S) \to \mathcal{S}\mathrm{hv}_{\mathrm{Nis}}(S)_*$ preserves motivic equivalences, filtered colimits (both by Lemma 1.9), and $\mathbb{P}^1$-loops, the result follows. □

We also note the following.

Lemma 1.11. *There are canonical equivalences*

$$L_{\mathrm{st,mot}}\mathcal{S}\mathrm{p}^{\mathbb{N}}(\mathcal{S}\mathrm{hv}_{\mathrm{Nis}}(S)_*, \mathbb{P}^1) \simeq \mathcal{SH}(S)$$

and

$$L_{\mathrm{st,mot}}\mathcal{S}\mathrm{p}^{\mathbb{N}}(\mathcal{S}\mathrm{hv}^{\mathrm{fr}}_{\mathrm{Nis}}(S), \gamma^*\mathbb{P}^1) \simeq \mathcal{SH}^{\mathrm{fr}}(S).$$

Proof. We prove the result for unframed spectra; the other case is similar. It is easy to see that $L_{\mathrm{mot}}\mathcal{S}\mathrm{p}^{\mathbb{N}}(\mathcal{S}\mathrm{hv}_{\mathrm{Nis}}(S)_*, \mathbb{P}^1) \simeq \mathcal{S}\mathrm{p}^{\mathbb{N}}(\mathcal{S}\mathrm{pc}(S)_*, \mathbb{P}^1)$ (see e.g. [Bac18, Lemma 26]). But $\mathbb{P}^1$ is symmetric in $\mathcal{S}\mathrm{pc}(S)_*$ [Hoy17, Lemma 6.3] and hence the result follows from Remark 1.2. □

Let $G : \mathcal{S}\mathrm{hv}_{\mathrm{Nis}}(S)_* \to \mathcal{S}\mathrm{hv}_{\mathrm{Nis}}(S)_*$ be an endofunctor. We say that G is *mixed prolongable* if we are given a natural transformation $\Sigma_{\mathbb{P}^1} G \to G\Sigma_T$. Then G naturally induces a functor

$$G : \mathcal{S}\mathrm{p}^{\mathbb{N}}(\mathcal{S}\mathrm{hv}_{\mathrm{Nis}}(S)_*, T) \to \mathcal{S}\mathrm{p}^{\mathbb{N}}(\mathcal{S}\mathrm{hv}_{\mathrm{Nis}}(S)_*, \mathbb{P}^1).$$

Let $G_n = \Omega^n_{\mathbb{P}^1}\Sigma^n_T$. This is mixed prolongable via

$$\Omega^n_{\mathbb{P}^1}\Sigma^n_T \xrightarrow{\Omega^n_{\mathbb{P}^1} u \Sigma^n_T} \Omega^n_{\mathbb{P}^1}\Omega_T\Sigma^{n+1}_T \xrightarrow{a^*} \Omega^{n+1}_{\mathbb{P}^1}\Sigma^{n+1}_T;$$

here $a : \mathbb{P}^1 \to T$ is the canonical map. For $X \in \mathrm{Sm}_S$, let $\Sigma^\infty_T X$ denote the associated T-suspension prespectrum. Then

$$G_0\Sigma^\infty_T X = (X, T \wedge X, T^2 \wedge X, \dots) \in \mathcal{S}\mathrm{p}^{\mathbb{N}}(\mathcal{S}\mathrm{hv}_{\mathrm{Nis}}(S)_*, \mathbb{P}^1)$$

is a spectrum motivically equivalent to $\Sigma^\infty_{\mathbb{P}^1} X$. By Corollary 1.10 and Lemma 1.11 it is hence enough to show that

$$G_0\Sigma^\infty_T X \to \gamma^{\mathbb{N}}_* \gamma^*_{\mathbb{N}} G_0 \Sigma^\infty_T X$$

is a stable motivic equivalence. There are canonical maps of mixed prolongable functors $G_0 \to G_1 \to \dots$, and one checks that

$$QG_0\Sigma^\infty_T X \simeq \operatorname*{colim}_i G_i \Sigma^\infty_T X.$$

In particular the map

$$G_0\Sigma^\infty_T X \to \operatorname*{colim}_i G_{3i-1}\Sigma^\infty_T X$$

is a stable equivalence.

The functor $\Omega^n_{\mathbb{P}^1}\Sigma^n_T$ is mixed prolongable in another way, using the lax module structure. Denote the mixed prolongable functor obtained in this way by $\tilde{G}_n$. Arguing as in Example 1.6, G_n and $\tilde{G}_n$ differ by cyclic permutations of $\mathbb{P}^1, T$ of order $n+1$. Note that the functor $\underline{\mathrm{Hom}}(-,-)$ preserves $\mathbb{A}^1$-homotopy equivalences in both variables. Since the cyclic permutation on $(\mathbb{P}^1)^{\wedge 3n}$ is $\mathbb{A}^1$-homotopic to the identity, and the same holds for T, we deduce that $G_{3i-1} \overset{\mathbb{A}^1}{\simeq} \tilde{G}_{3i-1}$ as prolongable functors. We learn that the canonical map

$$G_0\Sigma^\infty_T X \to \operatorname*{colim}_i G_{3i-1}\Sigma^\infty_T X \overset{\mathbb{A}^1}{\simeq} \operatorname*{colim}_i \tilde{G}_{3i-1}\Sigma^\infty_T X \simeq \operatorname*{colim}_i \tilde{G}_i \Sigma^\infty_T X$$

is an $\mathbb{A}^1$-equivalence.

Let E_i denote the sifted cocontinuous approximation of $\tilde{G}_i$, so that there is a map $E_i \to \tilde{G}_i$ of mixed prolongable functors. We can view h^{efr} (and h^{fr}) as mixed prolongable functors (note that they preserve Nisnevich equivalences in $\mathcal{P}_\Sigma(\mathrm{Sm}_{S+})$ by [EHK+19b, Propositions 2.3.7(ii) and 2.1.5(iii)] and so descend to Nisnevich sheaves) by using their lax module structures. By Voevodsky's Lemma, $E_i \simeq h^{\mathrm{efr},i}$ as lax modules and hence as mixed prolongable functors. Thus by the cone theorem (see Theorem 2.1 in §2), the map

$$E_i\Sigma^\infty_T X \to \tilde{G}_i\Sigma^\infty_T X$$

is a level motivic equivalence. We obtain the following commutative diagram

$$\begin{tikzcd} G_0\Sigma_T^\infty X \arrow[r, "L_{\mathrm{st}}"] \arrow[dr] & G_\infty \Sigma_T^\infty X \arrow[r, leftrightarrow, "L_{\mathbb{A}^1}"] & \tilde{G}_\infty \Sigma_T^\infty X & \\ & E_\infty \Sigma_T^\infty X \arrow[ur, "L_{\mathrm{mot}}"] \arrow[r, "\simeq"] & h^{\mathrm{efr}}\Sigma_T^\infty X \arrow[r, "L_{\mathrm{mot}}"] & h^{\mathrm{fr}}\Sigma_T^\infty X. \end{tikzcd}$$

All maps are the canonical ones; labels on the arrows denote the type of equivalence. The composite $G_0\Sigma_T^\infty X \to h^{\mathrm{fr}}\Sigma_T^\infty X \simeq \gamma_*^{\mathbb{N}}\gamma_{\mathbb{N}}^* G_0\Sigma_T^\infty X$ is the unit of adjunction. The diagram proves this unit is a stable motivic equivalence. This concludes the proof.

2. The cone theorem

Primary sources: [GNP18, Dru18].

The cone theorem is the computation of the motivic homotopy type $h^{\mathrm{efr}}(X/U)$ (the "framed cone") of an open immersion $U \hookrightarrow X$ where X is smooth.

Theorem 2.1. *Let k be an infinite perfect field, X a smooth affine k-scheme, and $U \subset X$ open. Then there is a canonical map*

$$h^{\mathrm{efr},n}(X/U) \to h^{\mathrm{efr},n}(X,U) \tag{1}$$

which is a motivic equivalence.

For now we work over an arbitrary base scheme S. We have already discussed Voevodsky's lemma that describes $h^{\mathrm{efr},n}(X)$ in terms of maps of pointed sheaves. In general we can describe the sections of the (pointed) sheaf

$$L_{\mathrm{Nis}}(X/U),$$

as follows. Define

$$Q(X,U)(T) = \{(Z,\phi) : \phi : (T)^h_Z \to X, \phi^{-1}(X \setminus U) = Z, Z \subset T \text{ is a closed subset}\},$$

which is pointed at $(\emptyset, \mathrm{can})$. Here T^h_Z denotes the henselization of T in Z. There is canonical map

$$Q(X,U) \to L_{\mathrm{Nis}}(X/U),$$

which, sends a section (Z,ϕ) over T to the map

$$T \simeq L_{\mathrm{Nis}}(T^h_Z \coprod_{T^h_Z \setminus Z} T \setminus Z) \xrightarrow{\phi} X/U.$$

Lemma 2.2. [EHK+19b, Proposition A.1.4] *The map $Q(X,U) \to L_{\mathrm{Nis}}(X/U)$ is an isomorphism.*

The presheaf of equationally framed correspondences of level n can be phrased in these terms. Let us elaborate on how this is done. Recall that we have n closed immersions $(\mathbb{P}^1)^{n-1} \hookrightarrow (\mathbb{P}^1)^n$ as the components of the "divisor at ∞" (so that $\bigcup (\mathbb{P}^1)^{\times n-1}$ is the divisor $\partial\mathbb{P}$). We then have the fiber sequence (in sets)

$$h^{\mathrm{efr},n}(X)(T) \to Q(\mathbb{A}^n \times X, \mathbb{A}^n_X \setminus 0_X)((\mathbb{P}^1)^{\times n} \times T) \to \prod_{1 \le i \le n} Q(\mathbb{A}^n \times X, \mathbb{A}^n_X \setminus 0_X)((\mathbb{P}^1)^{\times n-1} \times T).$$

Via Lemma 2.2, $h^{\mathrm{efr},n}(X)$ is isomorphic to

$$\underline{\mathrm{Hom}}_{\mathrm{Shv}_{\mathrm{Nis},\bullet}}((\mathbb{P}^1)^{\wedge n} \wedge (-)_+, T^{\wedge n} \wedge X_+).$$

2.1. **Relative equationally framed correspondences.** We elaborate on the discussion in §1.4.3. Throughout X is a smooth affine S-scheme and we have a cospan of S-schemes

$$Y \overset{i}{\hookrightarrow} X \overset{j}{\hookleftarrow} X \setminus Y (=: U),$$

where i is a closed immersion and j is its open complement. The presheaf of *relative equationally framed correspondence* $h^{\mathrm{efr},n}(X,U)$ is then defined via a similar formula:

$$h^{\mathrm{efr},n}(X,U)(T) \to Q(\mathbb{A}^n \times X, \mathbb{A}^n_X \setminus (0 \times Y))((\mathbb{P}^1)^{\times n} \times T) \to \prod_{1 \le i \le n} Q(\mathbb{A}^n \times X, \mathbb{A}^n_X \setminus (0 \times Y))((\mathbb{P}^1)^{\times n-1} \times T).$$

The next lemma follows from the above discussion.

Lemma 2.3. *There is a canonical isomorphism of sheaves of sets*

$$h^{\mathrm{efr},n}(X,U) \to \underline{\mathrm{Hom}}_{\mathrm{Shv}_{\mathrm{Nis},\bullet}}((\mathbb{P}^1)^{\wedge n} \wedge (-)_+, T^{\wedge n} \wedge (X/U)).$$

Explicitly, elements of $h^{\mathrm{efr},n}(X,U)(T)$ are described as (equivalence classes of) tuples

$$(Z,(\phi,g),W),$$

where

(1) $Z \hookrightarrow \mathbb{A}^n_T$ is a closed subscheme, finite over T,
(2) W is an étale neighborhood of Z in $\mathbb{A}^n_T$,
(3) $(\phi,g) : W \to \mathbb{A}^n \times X$ is a morphism such that

$$Z = (\phi,g)^{-1}(0\times Y) = \phi^{-1}(0) \cap g^{-1}(Y).$$

For example, suppose that $X = \mathbb{A}^1$ and $U = \mathbb{G}_m$. Then $h^{\mathrm{efr}}_n(\mathbb{A}^1,\mathbb{G}_m)$ is isomorphic to

$$\underline{\mathrm{Hom}}_{\mathrm{Shv}_{\mathrm{Nis},\bullet}}((\mathbb{P}^1)^{\wedge n} \wedge (-)_+, T^{\wedge n+1}).$$

Remark 2.4. The Z in the definition of $Q(X,U)$ is not required to be finite. However, in the definition of $h^{\mathrm{efr},n}(X,U)$, the Z appearing is a closed subset of both $(\mathbb{P}^1)^{\times n}$ and $\mathbb{A}^n$, so both proper and affine, hence finite.

We will also need the next presheaf:

Definition 2.5. Let $h^{\mathrm{efr},n}_{\mathrm{qf}}(X,U) \subset h^{\mathrm{efr}}(X,U)$ be the subpresheaf consisting of those $(Z,(\phi,g),W)$ where $\phi^{-1}(0) \to T$ is quasi-finite.

Example 2.6. In $h^{\mathrm{efr},1}(\mathbb{A}^1,\mathbb{G}_m)(k)$, we have the cycle $c = (Z = 0_k,(0,x),\mathbb{A}^1)$, where 0 indicates the constant function at zero, so we are considering the zero locus of the map

$$(0,x) : \mathbb{A}^1 \to \mathbb{A}^1 \times \mathbb{A}^1.$$

In this situation, $0^{-1}(0) = \mathbb{A}^1$ and hence is *not* quasi-finite over the base field, so $c \notin h^{\mathrm{efr},1}_{\mathrm{qf}}(\mathbb{A}^1,\mathbb{G}_m)(k)$. On the other hand $0^{-1}(0) \cap x^{-1}(0) = 0$, which restores the finiteness of Z, as needed. Generically, we should expect quasi-finiteness of $\phi^{-1}(0)$ — the only function we need to avoid in the above example is literally the constant function at zero.

The relevance of the quasi-finite version is the following

Construction 2.7. We have a map

$$h^{\mathrm{efr},n}(X) \to h^{\mathrm{efr},n}(X,U) \qquad (W,(\phi,g),Z) \mapsto (W,(\phi,g),Z_Y = \phi^{-1}(0)\cap g^{-1}(Y)),$$

which obviously factors as

$$h^{\mathrm{efr},n}(X) \to h^{\mathrm{efr},n}_{\mathrm{qf}}(X,U), \tag{2}$$

since $\phi^{-1}(0)$ is, in fact, finite. Now, consider the diagram

$$h^{\mathrm{efr},n}(X \textstyle\coprod U) \rightrightarrows h^{\mathrm{efr},n}(X),$$

$$(Z,(\phi,g),W) \mapsto ((Z,(\phi,g),W),(Z_X,(\phi_X,g_X),W_X)),$$

where $(Z_X,(\phi_X,g_X),W_X)$ is the component of $(Z,(\phi,g),W)$ over X. Denote the set-theoretic coequalizer of this diagram (taken sectionwise) by $\tau_{\le 0}h^{\mathrm{efr},n}(X/U)$. The map (2) then further factors as

$$\begin{tikzcd}
& h^{\mathrm{efr}}(X) \arrow[dl] \arrow[dr] & \\
\tau_{\le 0}h^{\mathrm{efr},n}(X/U) \arrow[rr, dashed, "p"] & & h^{\mathrm{efr},n}_{\mathrm{qf}}(X,U).
\end{tikzcd}$$

We can explicitly describe the sections of the presheaf $\tau_{\le 0}h^{\mathrm{efr},n}(X/U)$: if $T \in \mathrm{Sm}_S$, then $\tau_{\le 0}h^{\mathrm{efr},n}(X/U)(T)$ is the quotient of $h^{\mathrm{efr},n}(X)$ modulo the following equivalence relation

$$(W,(\phi,g),Z) \sim (W',(\phi',g'),Z'),$$

if and only if there exists $(W'',(\phi'',g''),Z'')$ such that $g'' : W'' \to U \subset X$, $W = W' \coprod W''$ up to refining the étale neighbourhoods, and $g = g' \coprod g''$.

Remark 2.8. We warn the reader that the canonical map $p : h^{\mathrm{efr},n}(X \coprod Y) \to h^{\mathrm{efr},n}(X) \times h^{\mathrm{efr}}(Y)$ is not an equivalence (unless $X = \emptyset$ or $Y = \emptyset$). It becomes so after applying $L_{\mathbb{A}^1}$ and letting $n \to \infty$ [EHK+19b, Remark 2.19], [GP18a, Theorem 6.4].

Lemma 2.9. *Suppose that S is regular. The map $\tau_{\le 0}h^{\mathrm{efr},n}(X/U) \to h^{\mathrm{efr},n}_{\mathrm{qf}}(X,U)$ is an L_{Nis}-equivalence.*

Proof. Let T be the henselization of a smooth S-scheme in a point. It suffices to show that $p(T)$ is both surjective and injective.

Surjectivity: Take $(Z, (\phi, g), W) \in h_{\mathrm{qf}}^{\mathrm{efr},n}(X, U)(T)$ and put $Z_0 = \phi^{-1}(0)$, so that $Z = Z_0 \cap g^{-1}(Y)$. By assumption, $Z_0 \to T$ is quasi-finite, and hence Zariski's main theorem for quasi-finite maps [Sta18, Tag 05K0] yields a factorization $Z_0 \hookrightarrow \overline{Z_0} \xrightarrow{\pi} T$, where the first map is an open immersion and the map π is finite. By [Sta18, Tag 04GJ] we may write

$$\overline{Z_0} = \overline{C_1} \coprod \cdots \coprod \overline{C_n} \coprod D,$$

where $\overline{C_i}$ is local and connected, $\overline{C_i} \to T$ is finite and $D \to T$ avoids the closed point. We have a similar decomposition $Z = B_1 \coprod \cdots \coprod B_m$ (no extra component since Z is finite), and we may assume without loss of generality that $B_i \subset \overline{C_i}$. Replacing W by $W \setminus (\overline{C_{m+1}} \cup \cdots \cup \overline{C_n} \cup D)$, we may assume that $m = n$ and $D = \emptyset$. Let $C_i = \overline{Z_i} \cap Z_0$. Then $C_i \subset \overline{C_i}$ is open, but $B_i \subset C_i$ so that C_i contains the closed point of the local scheme $\overline{C_i}$. It follows that $C_i = \overline{C_i}$ is finite over T, and hence so is Z_0. By [GNP18, Lemma 4.2], $C_i \to \mathbb{A}^n_T$ is a closed embedding, and hence so is $Z_0 \to \mathbb{A}^n_T$. We deduce that $(Z_0, (\phi, g), W) \in h^{\mathrm{efr},n}(X)$; clearly this defines a preimage of $(Z, (\phi, g), W)$ under p.

Injectivity: Consider two cycles

$$c = (Z, (\phi, g), W), c' = (Z', (\phi', g'), W').$$

Put $Z_1 = Z \cap g^{-1}(Y)$ and $Z_1' = Z' \cap g^{-1}(Y)$. Now suppose that $p(c) = p(c')$; in other words $Z_1 = Z_1'$ and there exists an étale neighborhood W'' refining W and W' such that $(\phi, g)|_{W''} = (\phi', g')|_{W''}$. We may write $Z = C \coprod D$, where $D \cap Z_1 = \emptyset$ and every component of C meets Z_1 (using again [Sta18, Tag 04GJ]). Shrinking W to remove D replaces c by a cycle with the same image in $\tau_{\leq 0} h^{\mathrm{efr},n}(X/U)$; we may thus assume that $D = \emptyset$. Now $\sigma : W''_Z \to Z$ is open and its image contains all closed points, so σ is surjective. Since every closed point of Z lifts along σ and σ is étale, it follows that σ admits a section [Sta18, Tags 04GJ and 04GK]. Thus, shrinking W'' if necessary, we may assume that it is an étale neighborhood of Z. Arguing the same way for Z' concludes the proof. □

2.1.1. *Quotients versus homotopy quotients.* The quotient X/U is given by the geometric realization of the following diagram in presheaves. In other words, it is given by the bar construction $\mathrm{Bar}^{\coprod}(X, U^{\coprod \bullet}, *)$.

$$X_+ \leftleftarrows (X \coprod U)_+ \leftleftarrows (X \coprod U \coprod U)_+ \leftleftarrows \cdots . \tag{3}$$

By definition (as sifted-colimit preserving extensions) we get $h^{\mathrm{efr}}(X/U)$ is the colimit of the simplicial diagram

$$h^{\mathrm{efr}}(X) \leftleftarrows h^{\mathrm{efr}}(X \coprod U) \leftleftarrows h^{\mathrm{efr}}(X \coprod U \coprod U) \leftleftarrows \cdots .. \tag{4}$$

We remark that the first two maps coincide with those from Construction 2.7. There is thus a canonical map

$$h^{\mathrm{efr},n}(X/U) \to \tau_{\leq 0} h^{\mathrm{efr},n}(X/U),$$

which witnesses 0-truncation of the resulting geometric realization.

Construction 2.10. Composing with the map from Construction 2.7, we get maps

$$h^{\mathrm{efr},n}(X/U) \to \tau_{\leq 0} h^{\mathrm{efr},n}(X/U) \to h_{\mathrm{qf}}^{\mathrm{efr},n}(X, U) \hookrightarrow h^{\mathrm{efr},n}(X, U),$$

The composite is the map in question in the cone theorem.

We now claim that the first map is an equivalence, i.e., $h^{\mathrm{efr},n}(X/U)$ is 0-truncated.

Construction 2.11. Let $\mathbf{efr}(X, U)(T)$ denote the following (1-)category (in fact, a poset):

- the objects are elements of $h^{\mathrm{efr}}(X)(T)$.
- there is a morphism
$$(Z, (\phi, g), W) \to (Z', (\phi', g'), W'),$$
if and only if there exists a decomposition $Z \coprod Z'' = Z'$, $g'|_{Z''}$ factors through $U \subset X$, and $(\phi', g')|_{W'^h_Z} = (\phi, g)|_{W^h_Z}$.

Lemma 2.12. *There is canonical equivalence*

$$|N_\bullet \mathbf{efr}(X, U)(T)| \simeq h^{\mathrm{efr}}(X/U)$$

Proof. For this proof we will abbreviate $(W,(\phi,g),Z)$ as (Z,Φ); as we manipulate these cycles what happens on the data of the étale neighborhood and defining functions will be clear. For each n, we have a map

$$N_n\mathbf{efr}(X,U)(T) \to h^{\mathrm{efr}}(X \coprod U^{\coprod n})(T),$$

given by

$$(Z_0,\Phi_0) \to \cdots \to (Z_n,\Phi_n) \mapsto (Z_0 \coprod (Z_1 \setminus Z_0) \coprod (Z_2 \setminus Z_1) \coprod (Z_n \setminus Z_{n-1}), \Phi_n).$$

On the other hand, if $(Z,\Phi) \in h^{\mathrm{efr}}(X \coprod U^{\coprod n})(T)$ we get cycles $\{Z_i', \Phi_i'\}_{i \geq 1}$ by pulling back along the various inclusions $\{\iota_i : U \hookrightarrow X \coprod U^{\coprod n}\}$ and also a cycle (Z_0, Φ_0) by pulling back along $X \hookrightarrow X \coprod U^{\coprod n}$. This defines an element $N_n\mathbf{efr}(X,U)(T)$ by setting $(Z_i,\Phi) = (Z_0 \coprod Z_1' \coprod \cdots Z_i', \Phi_i)$, with the maps determined. These maps induce mutual inverses of simplicial sets. ☐

Lemma 2.13. *The space $|N_\bullet\mathbf{efr}(X,U)(T)|$ is 0-truncated.*

Proof. Consider the subcategory

$$\mathbf{efr}(X,U)(T)^0 \subset \mathbf{efr}(X,U)(T),$$

consisting of those cycles (Z,Φ) such that no (nonempty) connected component of Z factors through U. Then $\mathbf{efr}(X,U)(T)^0$ is a category with no non-identity arrows, whence $|N_\bullet\mathbf{efr}(X,U)(T)^0|$ is 0-truncated. The inclusion $\mathbf{efr}(X,U)(T)^0 \to \mathbf{efr}(X,U)(T)$ admits a right adjoint (given by discarding all components of Z that factor through U), and hence induces an equivalence on classifying spaces. The result follows. ☐

It follows that the canonical map

$$h^{\mathrm{efr},n}(X/U) \to \tau_{\leq 0}h^{\mathrm{efr},n}(X/U),$$

is a sectionwise equivalence of spaces. Combining Lemmas 2.9, 2.12 and 2.13, we have proved the following result.

Theorem 2.14. *Let S be a regular base scheme. The map*

$$h^{\mathrm{efr},n}(X/U) \to h^{\mathrm{efr},n}_{\mathrm{qf}}(X,U),$$

is an L_{Nis}-equivalence.

2.2. **Moving into quasi-finite correspondences.** In order to complete the proof of the cone theorem, we will need

Theorem 2.15. *Let $S = Spec(k)$, where k is an infinite perfect field. The inclusion of presheaves*

$$h^{\mathrm{efr},n}_{\mathrm{qf}}(X,U) \hookrightarrow h^{\mathrm{efr},n}(X,U),$$

is an $L_{\mathbb{A}^1}$-equivalence.

This is a moving lemma in motivic homotopy theory.

Remark 2.16. In [GNP18], this moving lemma was discovered for $X = \mathbb{A}^n$ and $U = \mathbb{A}^n \setminus 0$ which suffices for the purposes of computing the framed motives of algebraic varieties. We will follow the treatment [Dru18] which performs the moving lemma for more general pairs.

For the rest of this section, we work over an infinite perfect field.

2.3. **Moving data.** We again fix X and use T to denote test schemes. For fixed d (for degree) and n (for rank), we have the presheaf[3]

$$\mathrm{Sect}_{d,n}(X) : T \mapsto H^0(X \times \mathbb{P}^n \times T, \mathcal{O}(d)^{\oplus n}).$$

Inside $\mathbb{P}^n$, we have the scheme $\mathcal{N} := \operatorname{Spec} k[T_1,\cdots,T_n]/(T_1,\cdots,T_n)^2 \subset \mathbb{P}^n$, which is a second-order thickening of the origin

$$\{0\} \subset \mathbb{A}^n \subset \mathbb{P}^n.$$

Now, in $\mathrm{Sect}_{d,n}(X)$, we have the subpresheaf

$$\Gamma_{d,n}(X) \subset \mathrm{Sect}_{d,n}(X) : T \mapsto \{(s_i) : s_i|_{X \times \mathcal{N} \times T} = x_i \cdot x_\infty^{d-1}\}.$$

We will let $d \to \infty$ and set

$$\Gamma_n(X) := \operatorname*{colim}_d \Gamma_{d,n}(X).$$

[3]In fact, it is represented by a scheme over $X \times \mathbb{P}^n$, but we will not need this.

Suppose that $\vec{s} = (s_1, \cdots, s_n) \in \Gamma_{n,d}(X)(k)$. We define $f_{\vec{s}} : \mathbb{A}^n \times X \to \mathbb{A}^n \times X$ by

$$f_{\vec{s}}(x_1, \cdots, x_n) = (s_1(x_1, \cdots, x_n)/x_\infty^d, \cdots, s_n(x_1, \cdots, x_n)/x_\infty^d).$$

If $(Z, (\phi, g), W) \in h^{\mathrm{efr}}(X, U)(T)$, then we define

$$\vec{s} \cdot (Z, (\phi, g), W) = ((f_{\vec{s}} \circ \phi)^{-1}(0) \cap g^{-1}(Y), (f_{\vec{s}} \circ (g, \phi)), W).$$

Remark 2.17. Our moving lemma will involve moving the framing $\phi : W \to \mathbb{A}^n$ to general position by means of the action $\vec{s}\cdot$ defined above, such that:

(1) the support of $\phi^{-1}(0)$ is unchanged,
(2) the trivialization of the normal bundle of $\phi^{-1}(0)$ in W is unchanged.

These conditions explain the need to consider $\mathcal{N}$. Namely, the conditions on $\Gamma_{d,n}(X)$ tell us that $f_{\vec{s}}$ agrees with the identity up to a second order thickening around the origin. For each d there is a distinguished section $\vec{x} = (x_i x_\infty^{d-1})$ and $f_{\vec{x}} = \mathrm{id}$. The appearance of $\mathbb{P}^n$ instead of $\mathbb{A}^n$ is a standard compactification trick — the k-vector spaces $H^0(X \times \mathbb{P}^n \times T, \mathcal{O}(d)^{\oplus n})$ are finite dimensional for each n, d.

We can use $\Gamma_{d,n}(X)$ to move sections around.

Construction 2.18. We have a map

$$\Gamma_n(X) \times h^{\mathrm{efr},n}(X, U) \xrightarrow{m} h^{\mathrm{efr},n}(X, U),\ (\vec{s}, \Phi) \mapsto \vec{s} \cdot \Phi.$$

We need to be able to draw paths in $\Gamma_n(X)$ with controlled properties. This is made precise by the next result, whose proof will be discussed in §2.5.

Lemma 2.19. *Let $T_1, \ldots, T_n \in \mathrm{Sm}_k$, $c_i \in h^{\mathrm{efr},n}(X, U)(T_i)$. There exists*

$$\gamma : \mathbb{A}^1 \to \Gamma_n(X)$$

such that

(1) $\gamma(0) = \vec{x}$,
(2) for each i, $\gamma(1) \cdot c_i \in h^{\mathrm{efr},n}_{\mathrm{qf}}(X, U) \subset h^{\mathrm{efr},n}(X, U)$, and
(3) for each j such that $c_j \in h^{\mathrm{efr},n}_{\mathrm{qf}}(X, U)(T_j)$, the composite

$$\mathbb{A}^1 \times T_j \xrightarrow{\gamma \times c_j} \Gamma_n(X) \times h^{\mathrm{efr},n}(X, U) \xrightarrow{m} h^{\mathrm{efr},n}(X, U)$$

factors through $h^{\mathrm{efr},n}_{\mathrm{qf}}(X, U) \subset h^{\mathrm{efr},n}(X, U)$.

2.4. **Filtration and finishing the proof.** Granting ourselves the above lemma, we finish the proof of the cone theorem.

Let $\alpha = \{(T_1, c_1), (T_2, c_2), \ldots, (T_n, c_n)\}$ be a finite collection of sections of $h^{\mathrm{efr},n}(X, U)$. Let $h^{\mathrm{efr},n}(X, U)^\alpha \subset h^{\mathrm{efr},n}(X, U)$ denote the subpresheaf generated by α, i.e. the image of the morphism of presheaves

$$h_{T_1} \coprod \cdots \coprod h_{T_n} \xrightarrow{c_1, \ldots, c_n} h^{\mathrm{efr},n}(X, U),$$

where h_{T_i} denotes the presheaf represented by T_i, and $\coprod$ denotes the coproduct of presheaves. Similarly denote by $h^{\mathrm{efr},n}_{\mathrm{qf}}(X, U)^\alpha \subset h^{\mathrm{efr},n}_{\mathrm{qf}}(X, U)$ the subpresheaf generated by those c_i with $c_i \in h^{\mathrm{efr},n}_{\mathrm{qf}}(X, U)$.

Lemma 2.20. *For any finite collection α as above there exists $\vec{s}$ such that the diagram of presheaves*

$$\begin{tikzcd}
\mathrm{Sing}^{\mathbb{A}^1} h^{\mathrm{efr,qf}}(X, S)^\alpha \arrow[r] \arrow[d] & \mathrm{Sing}^{\mathbb{A}^1} h^{\mathrm{efr,qf}}(X, U) \arrow[d, "\iota"] \\
\mathrm{Sing}^{\mathbb{A}^1} h^{\mathrm{efr}}(X, S)^\alpha \arrow[r] \arrow[ur, "\vec{s}\cdot"] & \mathrm{Sing}^{\mathbb{A}^1} h^{\mathrm{efr}}(X, U),
\end{tikzcd}$$

commutes up to simplicial homotopy. In particular we have a homotopy commutative diagram of spaces after geometric realization.

Proof. Since $\mathrm{Sing}^{\mathbb{A}^1}$ converts $\mathbb{A}^1$-homotopies to simplicial homotopies, it suffices to find a diagram with $\mathrm{Sing}^{\mathbb{A}^1}$ removed, and $\mathbb{A}^1$-homotopies filling the triangles. Apply Lemma 2.19 to the collection α to obtain $\gamma : \mathbb{A}^1 \to \Gamma_d(X)$. Let $\vec{s} = \gamma(1)$. Via Construction 2.18, γ yields a homotopy H : $\mathbb{A}^1 \times h^{\mathrm{efr},n}(X, U) \to h^{\mathrm{efr},n}(X, U)$, which starts at the identity by condition (1) of Lemma 2.19, and ends at the map $\vec{s}\cdot : h^{\mathrm{efr}}(X, S)^\alpha \to h^{\mathrm{efr,qf}}(X, U)$ by condition (2). The homotopy H restricts to a homotopy $\mathbb{A}^1 \times h^{\mathrm{efr},n}_{\mathrm{qf}}(X, U)^\alpha \to h^{\mathrm{efr},n}_{\mathrm{qf}}(X, U)$, by condition (3). This is what we need. $\square$

Proof of Theorem 2.15. Using Lemma 2.21 below, it suffices to solve the following lifting problem: given a map $K \to J$ of finite spaces and $T \in \mathrm{Sm}_k$, the following diagram admits a filler up to homotopy:

$$\begin{tikzcd} K \arrow[r] \arrow[d] & L_{\mathbb{A}^1} h_{\mathrm{qf}}^{\mathrm{efr},n}(X,U)(T) \arrow[d, "\iota"] \\ J \arrow[r] \arrow[ur, dashed] & L_{\mathbb{A}^1} h^{\mathrm{efr},n}(X,U)(T). \end{tikzcd}$$

By construction, $h^{\mathrm{efr},n}(X,U) \simeq \mathrm{colim}_\alpha\, h^{\mathrm{efr},n}(X,U)^\alpha$, and similarly for $h_{\mathrm{qf}}^{\mathrm{efr},n}(X,U)$, and moreover these colimits are filtered. Since $L_{\mathbb{A}^1}$ preserves colimits, colimits of presheaves are computed sectionwise, and K, J are compact objects of the ∞-category of spaces, we reduce to solving the following lifting problem

$$\begin{tikzcd} K \arrow[r] \arrow[d] & L_{\mathbb{A}^1} h_{\mathrm{qf}}^{\mathrm{efr},n}(X,U)^\alpha(T) \arrow[r] \arrow[d, "\iota"] & L_{\mathbb{A}^1} h_{\mathrm{qf}}^{\mathrm{efr},n}(X,U)(T) \arrow[d] \\ J \arrow[r] \arrow[urr, dashed] & L_{\mathbb{A}^1} h^{\mathrm{efr},n}(X,U)^\alpha(T) \arrow[r] & L_{\mathbb{A}^1} h_{\mathrm{qf}}^{\mathrm{efr},n}(X,U)(T). \end{tikzcd}$$

This follows from Lemma 2.20 which fills in in the right hand square. $\square$

We have used the following elementary observation

Lemma 2.21. *Let $f : X \to Y$ be a morphism of spaces. Then f is an equivalence if and only if for all maps of finite spaces $K \to J$ and all commutative diagrams of solid arrows,*

$$\begin{tikzcd} K \arrow[r] \arrow[d] & X \arrow[d] \\ J \arrow[r] \arrow[ur, dashed] & Y, \end{tikzcd}$$

a filler (indicated by the dashed arrows) exists which makes the diagram commute up to homotopy.

Proof. Clearly if f is an equivalence then a filler exists. Hence assume the filling condition. We will show that for all base points $x \in X$ and $n \geq 1$, the map

$$\pi_n(X, x) \to \pi_n(Y, f(x))$$

is an isomorphism, and that f induces an isomorphism on connected components; in other words f is an equivalence. To see this, we plug in different values for $K \to J$

(1) For $n \geq 1$, to show surjectivity, we plug in $* \to S^n$,
(2) and to show injectivity, we plug in $S^n \to *$,
(3) to show surjectivity on connected components we plug in $\emptyset \to *$,
(4) to show injectivity on connected components we plug in $S^0 \to *$.

$\square$

2.5. **Proof of Lemma 2.19.** We now prove the key moving lemma, following arguments of Druzhinin. We will in fact establish the following stronger result.

Theorem 2.22. *Let $T \in \mathrm{Sm}_k$ and $c \in h^{\mathrm{efr},n}(X,U)(T)$. There exists a path $\gamma : \mathbb{A}^1 \to \Gamma_n(X)$ with the following properties:*

(1) $\gamma(0) = \vec{x}$,
(2) $\gamma(1) \cdot c \in h_{\mathrm{qf}}^{\mathrm{efr},n}(X,U)(T)$, *and*
(3) given $S \in \mathrm{Sm}_k$ and $\varphi : S \to T$ such that $\varphi^*(c) \in h_{\mathrm{qf}}^{\mathrm{efr},n}(X,U)(S)$, the composite

$$\mathbb{A}^1 \times S \xrightarrow{\gamma \times \varphi^* c} \Gamma_n(X) \times h^{\mathrm{efr},n}(X,U) \xrightarrow{m} h^{\mathrm{efr},n}(X,U)$$

factors through $h_{\mathrm{qf}}^{\mathrm{efr},n}(X,U) \subset h^{\mathrm{efr},n}(X,U)$.

Lemma 2.19 follows from this by taking $T = \coprod_i T_i$, $c = \coprod_i c_i$, and $S = T_j \hookrightarrow T$ for the various T_j such that c_j is quasi-finite.

2.5.1. *Step A: preparations.* We now fix

$$c = (W, (\phi, g), Z) \in h^{\mathrm{efr}}(X, U)(T)$$

We denote by $\mathcal{E} := \mathbb{A}^n \times X \times T$, which comes equipped with a map from W

$$\psi := (\phi, g, \pi_T) : W \to \mathcal{E},$$

where $\pi_T : W \to \mathbb{A}^n_T \to T$. We can compactify $\mathcal{E}$ in the obvious way — define $\overline{\mathcal{E}} := \mathbb{P}^n \times X \times T$ so that we have a ample line bundle $\mathcal{O}(1)$ on $\overline{\mathcal{E}}$, pulled back from $\mathbb{P}^n$.

Now, choose an auxiliary m and consider a number of T-schemes:

(1) First, we have
$$\mathcal{E}^m_{\mathrm{disj}} \subset \mathcal{E}^{\times m}_U$$
whose fiber over $t \in T$ consists of those *distinct* m-tuples
$$((z_1, x_1), \cdots, (z_m, x_m)),$$
such that $(z_i, x_i) \notin 0 \times Y$.

(2) Next, define
$$W^m_{\mathrm{disj}} := (\psi^m)^{-1}(\mathcal{E}^m_{\mathrm{disj}}) \subset W^m_T;$$
explicitly, these are tuples $(w_1, \cdots, w_m)$ such that $(\phi, g)(w_i) \notin 0 \times Y$.

(3) Finally, we consider $\mathcal{C}^m$
$$\mathcal{C}^m \subset (\Gamma_{d,n}(X) \times T) \times_T W^m_{\mathrm{disj}}$$
consisting of those
$$\{(\vec{s}, (w_1, \cdots, w_m)) : \forall i\, f_{\vec{s}}((\phi, g))(w_i) \in 0 \times X\}.$$

We record the dimensions of $\mathcal{E}^m_{\mathrm{disj}}$ and W^m_{disj} which are consequences of the fact that they are open subschemes of smooth schemes.

Lemma 2.23. *We have*[4]

$$\dim_T \mathcal{E}^m_{\mathrm{disj}} = mn \cdot \dim X \qquad \dim_T W^m_{\mathrm{disj}} = mn.$$

2.5.2. *Step B: dimension estimate.* In this step, we compute the codimension of $\mathcal{C}^m$ fiber-wise, which will give us an estimate of the codimension of $\mathcal{C}^m$ in $(\Gamma_{d,n}(X) \times T) \times_T W^m_{\mathrm{disj}}$.

Lemma 2.24. *For each $(w_1, \cdots, w_m) \in W^m_{\mathrm{disj}}$ consider the fiber $\mathcal{C}^m_{(w_1,\cdots,w_m)} \subset \Gamma_{d,n}(X)$. Then for d large enough, we have that*

$$\mathrm{codim}_{\Gamma_{d,n}(X)} \mathcal{C}^m_{(w_1,\cdots,w_m)} = nm.$$

Proof. We note that k-points of the fiber are described as the subset of those $\vec{s}$ such that $f_{\vec{s}}((\phi, g))(w_i) = 0$ for all $i = 1, \cdots, m$. We need some notation:

(1) We have the canonical map $\psi^{\times m} : W^m_{\mathrm{disj}} \to \mathcal{E}^m_{\mathrm{disj}}$.

(2) we have projections:
$$p_1 : \mathcal{E}^m_{\mathrm{disj}} \to (\mathbb{P}^n \times X)^{\times m},$$
and
$$p_2 : \mathcal{E}^m_{\mathrm{disj}} \to T.$$

(3) $u = p_2(\psi(w_1), \cdots, \psi(w_m))$ is the point in T over which $(w_1, \cdots, w_m)$ live.

(4) $p_1(\psi(w_1), \cdots, \psi(w_m)) = ((z_1, x_1), \cdots, (z_m, x_m))$ so that $(\phi, g)(w_i) = (z_i, x_i)$.

(5) Consider the subscheme
$$\vec{W} := \{(z_i, x_i, u\}_i \subset \mathbb{P}^n \times X \times T.$$

To compute the codimension of interest, we consider the map

$$\Gamma_{d,n}(X)(k(u)) \to k(\vec{W})^n = k(u)^{nm}.$$

We note that the following sequence is an exact sequence of $k(u)$-vector spaces

$$0 \to \mathcal{C}^m_{(w_1,\cdots,w_m)} \to \Gamma_{d,n}(X)(k(u)) \to k(u)^{nm}.$$

The right most map is surjective by Lemma 2.25 below. We thus conclude that

$$\dim_{k(u)} \Gamma_{d,n}(X)(k(u)) - \dim_{k(u)} \mathcal{C}^m_{(w_1,\cdots,w_m)} = \dim_{k(u)} k(u)^{nm},$$

as desired. ☐

[4]Here, if X is T-scheme, we define $\dim_T X := \dim X - \dim T$.

Lemma 2.25. *In notation of the above proof, the map*

$$\Gamma_{d,n}(X)(k(u)) \to k(\vec{W})^n = k(u)^{nm},$$

$$(s_1, \cdots, s_n) \mapsto (f_{\vec{s}}(z_1, x_1), \cdots, f_{\vec{s}}(z_m, x_m)).$$

is surjective for $d \gg 0$.

Proof. The strategy is to find the above map as a specialization of a more global map.

(1) We have the projections (over T)

$$\mathrm{pr}_i : \mathcal{E}^m_{\text{disj}} \to \overline{\mathcal{E}},$$

for $i = 1, \cdots m$ and we have their graphs

$$\Delta_i \subset \mathcal{E}^m_{\text{disj}} \times_T \overline{\mathcal{E}}$$

(2) We have

$$\mathcal{E}^m_{\text{disj}} \times \mathcal{N} \times Y \cong \mathcal{E}^m_{\text{disj}} \times_T (T \times \mathcal{N} \times Y) \subset \mathcal{E}^m_{\text{disj}} \times_T \overline{\mathcal{E}}.$$

Consider the closed immersion whose ideal sheaf we denote by $\mathcal{I}$.

$$i : \coprod_{i=1}^m \Delta_i \coprod (\mathcal{E}^m_{\text{disj}} \times \mathcal{N} \times Y) \hookrightarrow \mathcal{E}^m_{\text{disj}} \times_T \overline{\mathcal{E}},$$

and consider the canonical map (we pulled back $\mathcal{O}_{\overline{\mathcal{E}}}$ to be over $\mathcal{E}^m_{\text{disj}} \times_T \overline{\mathcal{E}}$)

$$\mathcal{O}(d)^{\oplus n} \to i_* i^* \mathcal{O}(d)^{\oplus n},$$

which we push along the projection $r : \mathcal{E}^m_{\text{disj}} \times_U \overline{\mathcal{E}} \to \mathcal{E}^m_{\text{disj}}$. We have a map

$$r_* \mathcal{O}(d)^{\oplus n} \to r_* i_* i^* \mathcal{O}(d)^{\oplus n}, \tag{5}$$

whose kernel is given by $r_* \mathcal{I}(d)^{\oplus n}$. Now this map is surjective for $d \gg 0$ by Serre's vanishing criterion for ample line bundles [Sta18, Tag 0B5U] applied to $H^1(\mathcal{E}^m_{\text{disj}}, r_* \mathcal{I}(d)^{\oplus n})$.

Now consider the map (5):

(1) the left hand side is simply the constant sheaf on $H^0(\mathbb{P}^n \times X, \mathcal{O}(d))^{\oplus n}$, while
(2) the right hand side is

$$\oplus k[\mathcal{E}^m_{\text{disj}}]^n \oplus H^0(\mathcal{N} \times Y, \mathcal{O}(d))^{\oplus n}.$$

The result follows by the fact that taking stalks preserves surjections. □

2.5.3. *Step C: $\Gamma_{d,n}(X)$ has rational points for $d \gg 0$.* Now, we define the primary antagonist (sections that *do not* move c appropriately): let

$$\mathrm{Bad}_{d,n} \subset \Gamma_{d,n}(X) \times T,$$

be the subset

$$\{(\vec{s}, t) : (f_{\vec{s}} \circ \phi)^{-1}(0)|_t \text{ is not quasi-finite over } t\}$$

We also have $\mathrm{BadVanish}_{d,n}$ by

$$\mathrm{BadVanish}_{d,n} := \mathrm{Bad}_{d,n} \times_{\Gamma_{d,n}(X) \times T} \mathcal{C}^m \hookrightarrow \mathcal{C}^m$$

Lemma 2.26. *The fibers of the projection $\pi : \mathrm{BadVanish}_n \to \mathrm{Bad}_n$ have dimension $\geq m$.*

Proof. Suppose that $(\vec{s}, u) \in \mathrm{Bad}_n(k)$. Then the fiber of π over $(\vec{s}, u)$ consists of those m-tuples $(w_1, \cdots, w_m)$ such that $w_i \in (f_{\vec{s}} \circ (\phi, g)^{-1}(0)$ *not quasi-finite* for some $i = 1, \cdots, m$. Let $X \subset (f_{\vec{s}} \circ (\phi, g))^{-1}(0)$ be the non-quasi-finite locus; it must be of dimension ≥ 1. Any m distinct points of X contribute to the fiber, which must thus have dimension $\geq m$. The fibers over non-rational points are treated in the same way by base change. □

Lemma 2.27. *Consider the projection $q_d : \mathrm{Bad}_{d,n} \to \Gamma_{d,n}(X)$. For $d \gg 0$,*

$$\mathrm{codim}_{\Gamma_{d,n}(X)} \overline{q_d(\mathrm{Bad}_{d,n})} \geq m - \dim T.$$

Proof. For $d \gg 0$, Lemma 2.24 tells us that we have

$$\dim \mathcal{C}^m \leq \dim W_{\mathrm{disj}}^{\times m} + \dim \Gamma_{d,n}(X) - nm = \dim \Gamma_{d,n}(X) + \dim T,$$

where the last equality is Lemma 2.23.

The claim then follows using Lemma 2.26 by the estimates:

$$\begin{aligned} \dim \overline{q_d(\mathrm{Bad}_{d,n})} &\leq \dim \mathrm{Bad}_{d,n} \\ &\leq \dim \mathrm{BadVanish}_{d,n} - m \\ &\leq \dim \mathcal{C}^m - m \\ &\leq \dim \Gamma_{d,n}(X) + \dim T - m. \end{aligned}$$

□

2.5.4. *Step D: assembling the proof.*

Proof of Theorem 2.22. Let $m = \dim T + 2$. In this case, Lemma 2.27 shows that for $d \gg 0$:

$$\dim \overline{q_d(\mathrm{Bad}_{d,n})} \leq \dim T + \dim \Gamma_{d,n}(X) - m \leq \dim \Gamma_{d,n}(X) - 2.$$

Put

$$\mathcal{B}_c := \overline{q_d(\mathrm{Bad}_{d,n})}, \mathcal{U}_c := \Gamma_{d,n}(X) \setminus \mathcal{B}_c.$$

By construction, for any $\vec{s} \in \mathcal{U}_c$, we have $\vec{s} \cdot c \in h_{\mathrm{qf}}^{\mathrm{efr},n}(X,U)(T)$. We shall find a line in $\Gamma_{d,n}(X)$ starting at $\vec{x}$ and landing in $\mathcal{U}_c$. Consider the linear projection

$$\mathrm{pr}_{\vec{x}} : \Gamma_{n,d}(X) \setminus \vec{x} \to \mathbb{P}^N$$

where $N = \dim \Gamma_{d,n}(X)$. Note that $\dim \mathcal{B}_c \leq \dim \Gamma_{d,n}(X) - 2 = \dim \mathbb{P}^N - 1$. It follows that the closure of $\mathrm{pr}_{\vec{x}}(\mathcal{B}_c) \subset \mathbb{P}^N$ is a proper subset. Since k is infinite, the complement has a rational point, say L. Then L corresponds to a (rational) line in $\Gamma_{d,n}(X)$ through $\vec{x}$ which does not meet $\mathcal{B}_c$, except possibly in $\vec{x}$.

Let $\gamma : \mathbb{A}^1 \to \Gamma_{d,n}(X)$ correspond to this line. Properties (1) and (2) are clear by construction. For property (3), it suffices to show that if $c \in h_{\mathrm{qf}}^{\mathrm{efr},n}(X,U)(T)$, then the correspondence induced by γ over $\mathbb{A}^1 \times T$ is quasi-finite. Since this is a fiber-wise condition, it suffices to show that $\gamma(t) \cdot c \in h_{\mathrm{qf}}^{\mathrm{efr},n}(X,U)$ for all $t \in \mathbb{A}^1$. For $t \neq 0$ this holds by construction of L, and for $t = 0$ this holds by assumption.

This concludes the proof. □

3. The cancellation theorem

Primary sources: [EHK+19b, Voe10, AGP18].

3.1. **Group-complete framed spaces.**

Lemma 3.1 ([EHK+19b], Proposition 3.2.10(iii))**.** *The category $\mathcal{S}\mathrm{pc}^{\mathrm{fr}}(S)$ is semiadditive.*

It follows that, for every $\mathcal{X} \in \mathcal{S}\mathrm{pc}^{\mathrm{fr}}(S)$ and $X \in \mathrm{Sm}_S$, $\pi_0\mathcal{X}(X)$ is an abelian monoid.

Definition 3.2. We call $\mathcal{X}$ group-complete (or grouplike) if $\pi_0\mathcal{X}(X)$ is, for every $X \in \mathrm{Sm}_S$. We denote by $\mathcal{S}\mathrm{pc}^{\mathrm{fr}}(S)^{\mathrm{gp}} \subset \mathcal{S}\mathrm{pc}^{\mathrm{fr}}(S)$ the subcategory of group-complete spaces.

The group-complete spaces are closed under limits and filtered colimits, and hence the inclusion $\mathcal{S}\mathrm{pc}^{\mathrm{fr}}(S)^{\mathrm{gp}} \subset \mathcal{S}\mathrm{pc}^{\mathrm{fr}}(S)$ admits a left adjoint $\mathcal{X} \mapsto \mathcal{X}^{\mathrm{gp}}$ which is easily seen to be symmetric monoidal. The functor $\Omega^\infty : \mathcal{SH}(S) \simeq \mathcal{SH}^{\mathrm{fr}}(S) \to \mathcal{S}\mathrm{pc}^{\mathrm{fr}}(S)$ has image contained in $\mathcal{S}\mathrm{pc}^{\mathrm{fr}}(S)^{\mathrm{gp}}$. It follows that $\Sigma^\infty : \mathcal{S}\mathrm{pc}^{\mathrm{fr}}(S) \to \mathcal{SH}^{\mathrm{fr}}(S) \simeq \mathcal{SH}(S)$ inverts group completions and so factors through a symmetric monoidal, cocontinuous functor

$$\Sigma^\infty : \mathcal{S}\mathrm{pc}^{\mathrm{fr}}(S)^{\mathrm{gp}} \to \mathcal{SH}(S).$$

The following is the main result.

Theorem 3.3 ($\mathbb{P}^1$-cancellation)**.** *If k is a perfect field, then*

$$\Sigma^\infty : \mathcal{S}\mathrm{pc}^{\mathrm{fr}}(k)^{\mathrm{gp}} \to \mathcal{SH}(k)$$

is fully faithful.

Remark 3.4. The essential image of Σ^∞ is closed under colimits and known as the subcategory of *very effective spectra.*

Remark 3.5. The theorem is equivalent to showing that for $\mathcal{X}, \mathcal{Y} \in \mathcal{S}\mathrm{pc}^{\mathrm{fr}}(k)^{\mathrm{gp}}$ we have $\mathrm{Map}(\mathcal{X}, \mathcal{Y}) \simeq \mathrm{Map}(\Sigma_{\mathbb{P}^1}\mathcal{X}, \Sigma_{\mathbb{P}^1}\mathcal{Y})$, and this is further equivalent to showing that

$$\mathcal{Y} \to \Omega_{\mathbb{P}^1}\Sigma_{\mathbb{P}^1}\mathcal{Y}$$

is an equivalence. Here $\Sigma^1_{\mathbb{P}} : \mathcal{S}\mathrm{pc}^{\mathrm{fr}}(k)^{\mathrm{gp}} \to \mathcal{S}\mathrm{pc}^{\mathrm{fr}}(k)^{\mathrm{gp}}$ is the functor of tensor product with the image of $\mathbb{P}^1$ in $\mathcal{S}\mathrm{pc}^{\mathrm{fr}}(k)^{\mathrm{gp}}$.

Since $\mathbb{P}^1 \simeq S^1 \wedge \mathbb{G}_m$, it suffices to prove separate statements for these two suspensions. This is how we shall establish Theorem 3.3.

3.2. S^1-cancellation.

Proposition 3.6. *For $\mathcal{X} \in \mathcal{S}\mathrm{pc}^{\mathrm{fr}}(k)^{\mathrm{gp}}$, the canonical map*

$$\mathcal{X} \to \Omega_{S^1}\Sigma_{S^1}\mathcal{X}$$

is an equivalence.

Proof. Let $\mathcal{Y} = U\mathcal{X} \in \mathcal{P}_\Sigma(\mathrm{Corr}^{\mathrm{fr}}(S))$. We shall first compute $\Sigma_{S^1}\mathcal{Y}$. Let $X \in \mathrm{Sm}_S$. There is a finite coproduct preserving functor $c_X : \mathrm{Span}(\mathrm{Fin}) \to \mathrm{Corr}^{\mathrm{fr}}(S)$ sending $*$ to X. Its sifted-cocontinuous extension admits a right adjoint $c_{X*} : \mathcal{P}_\Sigma(\mathrm{Corr}^{\mathrm{fr}}(S)) \to \mathcal{P}_\Sigma(\mathrm{Span}(\mathrm{Fin})) \simeq \mathrm{CMon}(\mathcal{S}\mathrm{pc})$ [BH17, Proposition C.1] which preserves limits and sifted colimits, and hence all colimits by semiadditivity and [BH17, Lemma 2.8]. We deduce that

$$(\Sigma_{S^1}\mathcal{Y})(X) \simeq \Sigma_{S^1}(\mathcal{Y}(X)) \in \mathrm{CMon}(\mathcal{S}\mathrm{pc}). \tag{6}$$

This implies both that $\Sigma_{S^1}\mathcal{Y}$ is group-complete and, using that $\mathrm{CMon}(\mathcal{S}\mathrm{pc})^{\mathrm{gp}} \simeq \mathcal{SH}_{\geq 0}$ [Lur17, Remark 5.2.6.26], that

$$\mathcal{Y} \to \Omega_{S^1}\Sigma_{S^1}\mathcal{Y} \in \mathcal{P}_\Sigma(\mathrm{Corr}^{\mathrm{fr}}(S))^{\mathrm{gp}}$$

is an equivalence. To promote this to the same statement for $\mathcal{X} \in \mathcal{S}\mathrm{pc}^{\mathrm{fr}}(S)^{\mathrm{gp}}$, it is enough to show that $L_{\mathrm{Nis}}\Sigma_{S^1}\mathcal{Y}$ is motivically local; indeed Ω_{S^1} is computed sectionwise and hence preserves Nisnevich equivalences. Equation (6) shows that $\Sigma_{S^1}\mathcal{Y}$ is homotopy invariant; the result thus follows from Corollary 4.4 in §4. □

3.3. Abstract cancellation.

The following is extracted from [Voe10, §4].

Theorem 3.7. *Let $\mathcal{C}$ be a symmetric monoidal 1-category and $G \in \mathcal{C}$ a symmetric object. Suppose that the functor $\Sigma_G := G \otimes -$ admits a right adjoint Ω_G. Note that Ω_G is canonically a lax $\mathcal{C}$-module functor. Suppose that the unit transformation*

$$u : \mathrm{id}_{\mathcal{C}} \to \Omega_G\Sigma_G$$

admits a retraction ρ in the category of lax $\mathcal{C}$-module functors. Then u, ρ are inverse isomorphisms.

Remark 3.8. If $\mathcal{C}$ is an ∞-category and ρ is a lax $\mathcal{C}$-module retraction of $u : \mathrm{id}_{\mathcal{C}} \to \Omega_G\Sigma_G$, then the same conclusion holds (apply the theorem to $h\mathcal{C}$).

Remark 3.9. Since $\mathcal{C}$ is a 1-category, a lax $\mathcal{C}$-module structure on an endofunctor $F : \mathcal{C} \to \mathcal{C}$ just consists of compatible morphisms $X \otimes F(Y) \to F(X \otimes Y)$ for all $X, Y \in \mathcal{C}$. Moreover a transformation $\alpha : F \to G$ being a lax $\mathcal{C}$-module transformation is a property: it is the requirement that for $X, Y \in \mathcal{C}$, the following square commutes

$$\begin{tikzcd} X \otimes F(Y) \arrow[r, "\mathrm{id}_X \otimes \alpha_Y"] \arrow[d] & X \otimes G(Y) \arrow[d] \\ F(X \otimes Y) \arrow[r, "\alpha_{X \otimes Y}"] & G(X \otimes Y). \end{tikzcd}$$

Example 3.10. A lax $\mathcal{C}$-module transformation $\alpha : \mathrm{id} \to \mathrm{id}$ (of $\mathrm{id}_{\mathcal{C}}$ with its canonical $\mathcal{C}$-module structure) is completely determined by $\alpha_{\mathbb{1}} : \mathbb{1} \to \mathbb{1}$. In particular ρ being a retraction of u is equivalent to the composite

$$\mathbb{1} \xrightarrow{u_{\mathbb{1}}} \Omega_G G \xrightarrow{\rho_{\mathbb{1}}} \mathbb{1}$$

being the identity.

To simplify notation, from now on we will write $\underline{\mathrm{Hom}}(G, -)$ for Ω_G, and also use suggestive notation like $\otimes_{\mathrm{id}_Y} : \underline{\mathrm{Hom}}(A, B) \to \underline{\mathrm{Hom}}(A \otimes Y, B \otimes Y)$, when convenient.

Lemma 3.11. *For $X, Y \in \mathcal{C}$, the following diagram commutes*

$$\begin{tikzcd}
\underline{\mathrm{Hom}}(G, G \otimes X) \arrow[r, "\rho_X"] \arrow[d, "\otimes \mathrm{id}_Y"'] & \underline{\mathrm{Hom}}(\mathbb{1}, X) \arrow[d, "\otimes \mathrm{id}_Y"'] \\
\underline{\mathrm{Hom}}(G \otimes Y, G \otimes X \otimes Y) \arrow[r, "\Omega_Y \rho_{X \otimes Y}"] & \underline{\mathrm{Hom}}(Y, X \otimes Y).
\end{tikzcd}$$

Proof. Decompose the diagram as

$$\begin{tikzcd}
\underline{\mathrm{Hom}}(G, G \otimes X) \arrow[r, "\rho_X"] \arrow[d, "u"'] & \underline{\mathrm{Hom}}(\mathbb{1}, X) \arrow[d, "u"'] \\
\underline{\mathrm{Hom}}(Y, Y \otimes \underline{\mathrm{Hom}}(G, G \otimes X)) \arrow[r, "\underline{\mathrm{Hom}}(Y, Y \otimes \rho_X)"] \arrow[d] & \underline{\mathrm{Hom}}(Y, Y \otimes \underline{\mathrm{Hom}}(\mathbb{1}, X)) \arrow[d] \\
\underline{\mathrm{Hom}}(Y, \underline{\mathrm{Hom}}(G, G \otimes X \otimes Y)) \arrow[r, "\underline{\mathrm{Hom}}(Y, \rho_{X \otimes Y})"] \arrow[d, "\simeq"'] & \underline{\mathrm{Hom}}(Y, \underline{\mathrm{Hom}}(\mathbb{1}, X \otimes Y)) \arrow[d, "\simeq"'] \\
\underline{\mathrm{Hom}}(G \otimes Y, G \otimes X \otimes Y) \arrow[r, "\Omega_Y \rho_{X \otimes Y}"] & \underline{\mathrm{Hom}}(Y, X \otimes Y).
\end{tikzcd}$$

Here the middle vertical transformations are the lax module structure maps, and the bottom vertical isomorphisms hold in any symmetric monoidal category. The upper and lower squares commute by naturality, and the middle one by assumption of ρ being a lax module transformation. The vertical composites are given by $\otimes \mathrm{id}_Y$. This concludes the proof. □

Proof of Theorem 3.7. Let $X \in \mathcal{C}$. It suffices to show that the composite $\Omega_G \Sigma_G X \xrightarrow{\rho_X} X \xrightarrow{u_X} \Omega_G \Sigma_G X$ is the identity. Let $n \geq 2$ and $\alpha : G^{\otimes n} \to G^{\otimes n}$ be an automorphism. Consider the composite

$$p(\alpha) : \underline{\mathrm{Hom}}(G, G \otimes X) \xrightarrow{\mathrm{id}_{G^{\otimes n-1}} \otimes} \underline{\mathrm{Hom}}(G^{\otimes n}, G^{\otimes n} \otimes X) \xrightarrow{c_\alpha} \underline{\mathrm{Hom}}(G^{\otimes n}, G^{\otimes n} \otimes X) \xrightarrow{\rho^{n-1}} \underline{\mathrm{Hom}}(G, G \otimes X),$$

where c_α denotes the conjugation by α.

Note that the map "$\mathrm{id}_{G^{\otimes n-1}} \otimes$" is a composite of units u and hence by assumption of ρ being a retraction, we get $p(\mathrm{id}) = \mathrm{id}$.

On the other hand let $\alpha = \sigma$ be the cyclic permutation. Then the first $n-2$ applications of ρ are again "cancelling out identities", so that $\rho(\sigma)$ is the same as the composite

$$\underline{\mathrm{Hom}}(G, G \otimes X) \xrightarrow{f} \underline{\mathrm{Hom}}(G^{\otimes 2}, G^{\otimes 2} \otimes X) \xrightarrow{f_2} \underline{\mathrm{Hom}}(G, G \otimes X),$$

where f_1 "inserts id_G in the middle", and "f_2 applies ρ at the front". Lemma 3.11 implies that this is the same as $u_X \rho_X$.

Hence if G is n-symmetric, then since $\sigma = \mathrm{id}$ we find that

$$u_X \rho_X = p(\sigma) = p(\mathrm{id}) = \mathrm{id}\,.$$

This concludes the proof. □

3.4. **Twisted framed correspondences.** Using [EHK+19a, §B] it is possible to construct a symmetric monoidal ∞-category $\mathrm{Corr}^{\mathrm{fr}}_L(S)$ with the following properties:

- Its objects are pairs (X, ξ) with $X \in \mathrm{Sm}_S$ and $\xi \in K(X)$.
- The morphisms from (X, ξ) to (Y, ζ) are given by spans

  $$X \xleftarrow{f} Z \xrightarrow{g} Y,$$

  where Z is a *derived* scheme and f is a quasi-smooth morphism, together with a trivialization

  $$f^*(\xi) + L_f \simeq g^*(\eta) \in K(Z).$$

- There is a symmetric monoidal functor $\delta : \mathrm{Corr}^{\mathrm{fr}}(S) \to \mathrm{Corr}^{\mathrm{fr}}_L(S)$ which sends X to $(X, 0)$ and induces the evident maps on mapping spaces.

It follows that the tensor product in $\mathrm{Corr}^{\mathrm{fr}}_L(S)$ is given by the product of schemes, and the functor δ is faithful (induces monomorphisms on mapping spaces).

The following will be helpful.

Lemma 3.12. *A span*
$$X \xleftarrow{f} Z \to Y \in \mathrm{Map}_{\mathrm{Corr}^{\mathrm{fr}}_L(S)}((X,0),(Y,0))$$
is in the image of δ if and only if f is finite.

Proof. The only concern is that Z might be a derived scheme instead of a classical one; by [EHK$^+$19a, Lemma 2.2.1] this cannot happen. □

We mainly introduce the category $\mathrm{Corr}^{\mathrm{fr}}_L(S)$ for the following technically convenient reason: all of its objects are strongly dualizable.

Proposition 3.13. *Let $X \in \mathrm{Sm}_S$. The spans*
$$* \leftarrow X \xrightarrow{\Delta} X \times X$$
and
$$X \times X \xleftarrow{\Delta} X \to *$$
admit evident framings, and exhibit (X, L_X) as the dual of $(X,0)$ in $\mathrm{Corr}^{\mathrm{fr}}_L(S)$.

Proof. This kind of duality happens in all span categories; we just need to verify that the spans are frameable and that the induced framings of the compositions are trivial. All of this is easy to verify. For example $X \times X$ really means $(X,0) \otimes (X, L_X) = (X \times X, p_2^* L_X)$ and hence to frame the first span we need to exhibit a path
$$0 + L_X \simeq \Delta^* p_2^* L_X,$$
but this holds on the nose since $\Delta^* p_2^* \simeq \mathrm{id}$; to frame the second span we need to exhibit a path
$$\Delta^* p_2^* L_X + L_\Delta \simeq 0$$
which is possible in K-theory since the composite $X \xrightarrow{\Delta} X \times X \xrightarrow{p_1} X$ is the identity, so $0 = L_{\mathrm{id}} \simeq L_\Delta + \Delta^* L_{p_1}$ and finally $L_{p_1} \simeq p_2^* L_X$ by base change. □

The following will be helpful later to exhibit spans.

Construction 3.14. Suppose given $X, G \in \mathrm{Sm}_S$, a map $f : X \times G \to \mathbb{A}^1$ and a path $L_G \simeq 1 \in K(G)$. Then there is a span
$$D(f) : X \xleftarrow{p_1} Z(f) \xrightarrow{p_2} G \in \mathrm{Map}_{\mathrm{Corr}^{\mathrm{fr}}_L(S)}((X,0),(G,0));$$
the framing is given by
$$L_{p_1} \simeq L_{Z(f)/X \times G} + L_{X \times G/X} \simeq -1 + L_G \simeq 0 \in K(Z(f)),$$
where we have used that $L_{Z(f)/X \times G} \simeq -1$ via f and $L_G \simeq 1$ by assumption.

We will always apply this construction with $G = \mathbb{A}^1 \setminus 0$, so that there is a canonical trivialization of L_G.

3.5. **A general construction.** Given $X, Y \in \mathrm{Sm}_S$, for notational convenience we will write $f : X \rightsquigarrow Y$ for $f \in \mathrm{Map}_{\mathrm{Corr}^{\mathrm{fr}}_L(S)}((X,0),(Y,0))$.

Construction 3.15. Let $A, G \in \mathrm{Sm}_S$ and $\alpha : A \times G \rightsquigarrow G$. We obtain a $\mathrm{Corr}^{\mathrm{fr}}_L(S)$-module transformation
$$\rho_\alpha : \Omega_G \Sigma_G \to \Omega_A \in \mathrm{End}(\mathcal{P}_\Sigma(\mathrm{Corr}^{\mathrm{fr}}_L(S)))$$
as follows: via strong dualizability (Proposition 3.13), we can rewrite the source and target and consider the transformation
$$G^\vee \otimes G \otimes - \xrightarrow{\alpha^\vee \otimes \mathrm{id}_-} A^\vee \otimes -$$
where $\alpha^\vee : G^\vee \otimes G \to A^\vee$ is obtained from α in the evident manner.

We will eventually apply this with $G = \mathbb{A}^1 \setminus 0$ and $A = \mathbb{A}^1$ or $A = *$.

Remark 3.16. Let $X, Y \in \mathrm{Sm}_S$. Given a span
$$G \times Y \leftarrow Z \to G \times X,$$
the transformation ρ_α produces a span
$$A \times Y \leftarrow \rho_\alpha(Z) \to X.$$
Write α as
$$A \times G \leftarrow C \to G.$$

Tracing through the definitions, one finds that

$$\rho_\alpha(Z) = Z \times_{G \times G} C,$$

with an evident induced framing.

Lemma 3.17. *The transformation ρ_α satisfies the following properties.*

(1) Given $Z' : X \rightsquigarrow X'$ and $Z : G \times Y \rightsquigarrow G \times X$ we have

$$\rho_\alpha((\mathrm{id}_G \otimes Z') \circ Z) \simeq (\mathrm{id}_A \otimes Z') \circ \rho_\alpha(Z).$$

(2) Given $Z' : Y \rightsquigarrow Y$ and $Z : G \times Y \rightsquigarrow G \times X$ we have

$$\rho_\alpha(Z \circ (\mathrm{id}_G \otimes Z')) \simeq \rho_\alpha(Z) \circ (\mathrm{id}_G \otimes Z').$$

(3) Given $i : A' \rightsquigarrow A$, we have

$$\rho_{i^*\alpha} \simeq i^* \rho_\alpha.$$

Proof. Evident from the naturality of the construction. □

Now define

$$M_\alpha(Y, X) \subset \mathrm{Map}_{\mathrm{Corr}^{\mathrm{fr}}(S)}(G \times Y, G \times X)$$

to consist of the disjoint union of those path components corresponding to spans $G \times Y \leftarrow Z \rightarrow G \times X$ such that $\rho_\alpha(Z)$ is finite. Then (1) and (2) of Lemma 3.17 translate (using Lemma 3.12) into

(1) $(\mathrm{id}_G \otimes Z') \circ M_\alpha(Y, X) \subset M_\alpha(Y, X')$, and
(2) $M_\alpha(Y, X) \circ (\mathrm{id}_G \otimes Z') \subset M_\alpha(Y', X)$.

Construction 3.18. Define a subfunctor

$$F_\alpha \Omega_G \Sigma_G \hookrightarrow \Omega_G \Sigma_G \in \mathrm{End}(\mathcal{P}_\Sigma(\mathrm{Corr}^{\mathrm{fr}}(S)))$$

via

$$(F_\alpha \Omega_G \Sigma_G X)(Y) = M_\alpha(X, Y).$$

The lax monoidal natural transformation

$$\Omega_G \delta_* \delta^* \Sigma_G \simeq \delta_* \Omega_G \Sigma_G \delta^* \xrightarrow{\rho_\alpha} \delta_* \Omega_A \delta^* \simeq \Omega_A \delta_* \delta^*$$

restricts by construction to a natural transformation

$$\rho_\alpha : F_\alpha \Omega_G \Sigma_G \to \Omega_A,$$

which we will think of as

$$\rho_\alpha : A \otimes F_\alpha \Omega_G \Sigma_G \to \mathrm{id}.$$

Take $A = \mathbb{A}^1$, $G = \mathbb{A}^1 \setminus 0$ and suppose that $\rho_\alpha(\mathrm{id}_G)$ is finite. Then the unit transformation

$$\mathrm{id} \to \Omega_G \Sigma_G$$

factors through $F_\alpha \Omega_G \Sigma_G$. Moreover we obtain two $\mathbb{A}^1$-homotopic transformations

$$\rho_{i_0^*\alpha}, \rho_{i_1^*\alpha} : F_\alpha \Omega_G \Sigma_G \to \mathrm{id} \in \mathrm{End}(\mathcal{P}_\Sigma(\mathrm{Corr}^{\mathrm{fr}}(S))).$$

3.6. **$\mathbb{G}_m$-cancellation.** Let $G = \mathbb{A}^1 \setminus 0$.

Definition 3.19. We define maps $G \times G \to \mathbb{A}^1$ via

$$g_n^+(t_1, t_2) = t_1^n + 1 \text{ and } g_n^-(t_1, t_2) = t_1^n + t_2.$$

We further define maps $\mathbb{A}^1 \times G \times G \to \mathbb{A}^1$ via

$$h_n^\pm(t, t_1, t_2) = t g_n^\pm(t_1, t_2) + (1 - t) g_m^\pm(t_1, t_2).$$

Recall the associated spans from Construction 3.14. Put

$$F_i = \bigcap_{m,n \geq i} [F_{D(h_{m,n}^+)} \cap F_{D(h_{m,n}^-)}] \subset \Omega_G \Sigma_G.$$

Lemma 3.20. *We have*

$$\operatorname*{colim}_i F_i \simeq \mathrm{Map}_{\mathrm{Corr}^{\mathrm{fr}}(S)}(X, Y).$$

Proof. We follow [Voe10, Lemma 4.1 and Remark 4.2]. Suppose given $Y \leftarrow Z \rightarrow X \in \mathrm{Map}_{\mathrm{Corr}^{\mathrm{fr}}(S)}(Y, X)$. We shall exhibit an integer N such that for all $m, n > N$ the projection $Z' = \rho_{D(h^{\pm}_{m,n})}(Z) \rightarrow Y \times \mathbb{A}^1$ is finite; this will prove what we want. Write $f_1, f_2 : Z \rightarrow G$ for the two projections. Using Zariski's main theorem, we can form a commutative diagram

$$\begin{tikzcd} Z \arrow[r] \arrow[d, "f_1 \times p_Y"'] & \bar{C} \arrow[d, "\bar{f}_1 \times p_Y"'] \\ G \times Y \arrow[r] & \mathbb{P}^1 \times Y, \end{tikzcd}$$

where $\bar{f}_1 \times p_Y$ is finite. There exists N such that the rational function $\bar{f}_1^N / f_2$ is regular in a neighbourhood U_0 of $\bar{f}_1^{-1}(0)$ and $f_2/\bar{f}_1^N$ is regular in a neighbourhood U_∞ of $\bar{f}_1^{-1}(\infty)$. We have the function $h^{\pm} = h^{\pm}_{m,n}(t, f_1, f_2)$ on $Z \times \mathbb{A}^1$, and Remark 3.16 implies that $Z' = Z(h) \subset Z \times \mathbb{A}^1$. The composite $\bar{C} \times \mathbb{A}^1 \rightarrow \mathbb{P}^1 \times Y \times \mathbb{A}^1 \rightarrow Y \times \mathbb{A}^1$ is projective, and $Z(h) \rightarrow Y \times \mathbb{A}^1$ is affine. We will finish the proof by showing that $i^{\pm} : Z(h) \rightarrow \bar{C} \times \mathbb{A}^1$ is a closed immersion for $n, m > N$; indeed then $Z(h) \rightarrow Y \times \mathbb{A}^1$ will be both proper and affine, and hence finite as desired.

Note that h^+ extends to the regular map $t\bar{f}_1^m + (1-t)\bar{f}_1^n + 1 : \bar{C} \rightarrow \mathbb{P}^1$, which does not vanish if $\bar{f}_1 \in \{0, \infty\}$. Thus i^+ is always a closed immersion.

Now we deal with i^-. Let $U_1 = \bar{f}_1^{-1}(G)$. A morphism being a closed immersion is local on the target [Sta18, Tag 01QO], so it is enough to show that i is a closed immersion over U_0, U_∞ and U_1. This is clear for U_1. Consider the function $h_0 = t\bar{f}_1^n/f_2 + (1-t)\bar{f}_1^m/f_2 + 1$. By construction, this is regular on h_0, so $Z(h_0) \subset U_0$ is closed. Also by construction, $h_0 = 1$ if $\bar{f}_1 = 0$, and $h^- = f_2 h_0$ on $U_0 \setminus 0$, where f_2 is a unit. It follows that $Z(h_0) = U_0 \cap Z(h)$. A similar argument works for U_∞. □

Using Construction 3.18, we thus obtain a sequence of lax module transformations

$$\begin{tikzcd} \mathrm{id} \arrow[r, dashed, shift left] & F_0 \arrow[l, bend left, "\rho_0^{\pm}"] \arrow[r, hook] & F_1 \arrow[ll, bend right, "\rho_1^{\pm}"'] \arrow[r, hook] & \dots \arrow[lll, bend right=40, "\rho_2^{\pm}"'] \arrow[r, hook] & \Omega_G \Sigma_G, \end{tikzcd}$$

where the arrows to the right form a colimit diagram. The dashed arrow might not exist, but the lemma above implies that its composite sufficiently far to the right does, and this is all we need.[5] For $m \geq n$, the $h^{\pm}_{m,n}$ induce $\mathbb{A}^1$-homotopies making the following diagram commute

$$\begin{tikzcd} F_n \arrow[r, hook] \arrow[d, "\rho_n^{\pm}"] & F_m \arrow[dl, "\rho_m^{\pm}"] \\ \mathrm{id} & \end{tikzcd}$$

Applying $L_{\mathbb{A}^1}$, there are thus induced transformations on the colimit

$$\begin{tikzcd} L_{\mathbb{A}^1} \arrow[r, "u"] & L_{\mathbb{A}^1} \Omega_G \Sigma_G. \arrow[l, bend left, "\rho^{\pm}"] \end{tikzcd}$$

After group completion, we may take the difference, and hence obtain

$$\rho = \rho^+ - \rho^- : L^{\mathrm{gp}}_{\mathbb{A}^1} \Omega_G \Sigma_G \rightarrow L^{\mathrm{gp}}_{\mathbb{A}^1}.$$

We are now ready to prove our main result.

Theorem 3.21. *Let k be an infinite perfect field. Then the unit transformation*

$$u : \mathrm{id} \rightarrow \Omega_{\mathbb{G}_m} \Sigma_{\mathbb{G}_m} \in \mathrm{End}(\mathcal{S}\mathrm{pc}^{\mathrm{fr}}(k)^{\mathrm{gp}})$$

is an equivalence.

Proof. We seek to apply the abstract cancellation Theorem 3.7 (in the guise of Remark 3.8). Note that $\mathbb{G}_m$ is symmetric in $\mathcal{S}\mathrm{pc}^{\mathrm{fr}}(k)^{\mathrm{gp}}$: $T \simeq S^1 \wedge \mathbb{G}_m$ is symmetric by the usual argument, and S^1 is (symmetric and) semi-invertible (by S^1-cancellation, i.e. Proposition 3.6). We have already constructed a lax module transformation

$$\rho : L^{\mathrm{gp}}_{\mathrm{mot}} \Omega_G \Sigma_G \rightarrow L^{\mathrm{gp}}_{\mathrm{mot}}.$$

Corollary 4.5 in §4 shows that $L^{\mathrm{gp}}_{\mathrm{mot}} \Omega_G \Sigma_G \simeq \Omega_G \Sigma_G L^{\mathrm{gp}}_{\mathrm{mot}}$ and hence we obtain a lax module transformation

$$\rho : \Omega_G \Sigma_G \rightarrow \mathrm{id} \in \mathrm{End}(\mathcal{S}\mathrm{pc}^{\mathrm{fr}}(k)^{\mathrm{gp}}).$$

[5] One may verify that the arrow actually does exist.

In $\mathcal{S}\mathrm{pc}^{\mathrm{fr}}(k)^{\mathrm{gp}}$ there is a splitting $G \simeq \mathbb{1} \oplus \mathbb{G}_m$, and hence a retraction $\mathbb{G}_m \to G \to \mathbb{G}_m$. This induces a retraction of lax module functors

$$\Omega_{\mathbb{G}_m}\Sigma_{\mathbb{G}_m} \to \Omega_G\Sigma_G \to \Omega_{\mathbb{G}_m}\Sigma_{\mathbb{G}_m},$$

which in particular allows us to build the lax module transformation

$$\rho' : \Omega_{\mathbb{G}_m}\Sigma_{\mathbb{G}_m} \to \Omega_G\Sigma_G \to \mathrm{id}\,.$$

In order to apply the abstract cancellation theorem, it remains to verify that $\rho' u \simeq \mathrm{id}$. Via Example 3.10, for this it suffices to compute the effect of $\rho' u$ on $\mathrm{id}_{\mathbb{1}}$. Now $u(\mathrm{id}_{\mathbb{1}}) = \mathrm{id}_{\mathbb{G}_m}$, which corresponds to $\mathrm{id}_G - p \in \mathrm{Hom}(G, G)$, where $p : G \to * \to G$, and so $\rho' u(\mathrm{id}_{\mathbb{1}}) = \rho(\mathrm{id}_G) - \rho(p)$. The result thus follows from Lemma 3.22 below. □

Lemma 3.22. *For each $n > 0$ we have*

(1) $\rho_n^+(p) = \rho_n^-(p)$, and

(2) $\rho_n^+(\mathrm{id}_G) \overset{\mathbb{A}^1}{\simeq} \rho_n^-(\mathrm{id}_G) + \mathrm{id}_{\mathbb{1}}$.

Proof. This is essentially [Voe10, Lemma 4.3].

Note that p is represented by the correspondence $G \xleftarrow{\simeq} G \xrightarrow{1} G$, so that by Remark 3.16, $\rho_n^{\pm}(p)$ is represented by $Z(g_n^{\pm}(t, 1)) \to *$. But $g_n^+(t, 1) = g_n^-(t, 1)$, whence (1).

Similarly $\rho_n^{\pm}(\mathrm{id}_G)$ is represented by $Z_{\pm} := Z(g_n^{\pm}(t, t))$, so $Z_+ = Z(t^n + 1)$ and $Z_- = Z(t^n + t)$, where both $t^n + 1, t^n + t$ are viewed as functions on $\mathbb{A}^1 \setminus 0$. Consider $H = D(t^n + ts + 1 - s) : \mathbb{A}^1 \rightsquigarrow *$, where we view h as a function $\mathbb{A}^1 \times \mathbb{A}^1 \to \mathbb{A}^1$. Then H provides an $\mathbb{A}^1$-homotopy between $D(t^n + 1)$ and $D(t^n + t)$, where this time we view $t^n + 1, t^n + t$ as functions on $\mathbb{A}^1$. Now

$$Z(t^n + 1|\mathbb{A}^1) = Z(t^n + 1|\mathbb{A}^1 \setminus 0) = Z^+,$$

whereas

$$Z(t^n + t|\mathbb{A}^1) = Z(t^n + t|\mathbb{A}^1 \setminus 0) \coprod \{0\} = Z^- \coprod \{0\}.$$

Since $0 \subset \mathbb{A}^1 \to *$ defines the identity correspondence, H provides the desired homotopy.

This concludes the proof. □

4. Strict homotopy invariance

Primary sources: [GP18b, DP18].

The title of this section derives from the following. Write $\mathcal{P}_\Sigma(\mathrm{Sm}_k, \mathrm{Ab})$ for the category of additive presheaves of abelian groups on Sm_k.

Definition 4.1. Let $F \in \mathcal{P}_\Sigma(\mathrm{Sm}_k, \mathrm{Ab})$. Then F is called *homotopy invariant* (or $\mathbb{A}^1$-*invariant*) if for all $X \in \mathrm{Sm}_k$, the canonical map $F(X) \to F(X \times \mathbb{A}^1)$ is an isomorphism.

If F is a sheaf in the Nisnevich topology, then F is called *strictly homotopy invariant* if for all $n \geq 0$ and all $X \in \mathrm{Sm}_k$ the canonical map $H^n_{\mathrm{Nis}}(X, F) \to H^n_{\mathrm{Nis}}(X \times \mathbb{A}^1, F)$ is an isomorphism.

There are two important observations regarding this:

(1) If F is a homotopy invariant presheaf, it need not be the case that $a_{\mathrm{Nis}}F$ is homotopy invariant (let alone strictly homotopy invariant).
(2) If F is a homotopy invariant sheaf, it need not be strictly homotopy invariant.

However, it turns out that in the presence of transfers, neither of these problems occurs. The first general results in this direction were probably obtained by Voevodsky in [Voe00]. Here is a version for framed presheaves.

Theorem 4.2. *Let k be an infinite field, and $F \in \mathcal{P}_\Sigma(\mathrm{Corr}^{\mathrm{fr}}(k), \mathrm{Ab})$. Suppose that F is homotopy invariant.*

(1) The restriction of F to the small Nisnevich site of $\mathbb{A}^1$ is a sheaf.
(2) The sheafification $a_{\mathrm{Nis}}F$ is homotopy invariant.
(3) If k is perfect, then $a_{\mathrm{Nis}}F$ is strictly homotopy invariant.

Proof. Note that $\mathcal{P}_\Sigma(\mathrm{Corr}^{\mathrm{fr}}(k), \mathrm{Ab}) \simeq \mathcal{P}_\Sigma(h\mathrm{Corr}^{\mathrm{fr}}(k), \mathrm{Ab})$. There is a functor $\lambda : \mathrm{Corr}^{\mathrm{efr}}_*(k) \to h\mathrm{Corr}^{\mathrm{fr}}(k)$; see [EHK+19b, §3.4.7] for both the definiton of the functor λ and the category $\mathrm{Corr}^{\mathrm{efr}}_*(k)$. What matters for us is that if $F \in \mathcal{P}_\Sigma(h\mathrm{Corr}^{\mathrm{fr}}(k), \mathrm{Ab})$ then $\lambda^* F$ is a "stable, radditive" homotopy invariant presheaf with equationally framed transfers. The analogous result for these presheaves is established in [GP18b, DP18]. □

Remark 4.3. We believe that instead of arguing via the reduction to equationally framed correspondences, it should be relatively straightforward to adapt the arguments to tangentially framed correspondences directly. In fact the arguments for tangentially framed correspndences are likely *simpler* than for equationally framed ones. The clearest account of these arguments that we know is [DK18]; unfortunately we did not have time to check if all arguments translate directly.

We can escalate the above result as follows.

Corollary 4.4. *Let k be an infinite perfect field, and $F \in \mathcal{P}_\Sigma(\mathrm{Corr}^{\mathrm{fr}}(k))^{\mathrm{gp}}$ be $\mathbb{A}^1$-invariant. Then $L_{\mathrm{Nis}}F$ is $\mathbb{A}^1$-invariant, and hence motivically local.*

Proof. By an induction on the Postnikov tower, or equivalently using the (strongly convergent) descent spectral sequence, this is immediate from Theorem 4.2. □

We can also deduce the following fact, which is very important for the cancellation theorem.

Corollary 4.5. *Let k be an infinite perfect field. On the category $\mathcal{P}_\Sigma(\mathrm{Corr}^{\mathrm{fr}}(k))^{\mathrm{gp}}$, the canonical transformation $L_{\mathrm{mot}}\Omega_{\mathbb{G}_m} \to \Omega_{\mathbb{G}_m} L_{\mathrm{mot}}$ is an equivalence.*

Proof. Since this is a morphism of motivically local spaces, it suffices to prove that the map induces an equivalence on sections over fields [Mor05, Lemma 6.1.3]. Thus let K/k be a field extension. By Corollary 4.4, $L_{\mathrm{mot}} = L_{\mathrm{Nis}} L_{\mathbb{A}^1}$. Note that $\Omega_{\mathbb{G}_m}$ commutes with $L_{\mathbb{A}^1}$ (see e.g. [Bac19, Lemma 4]) and fields are stalks for the Nisnevich topology; hence it is enough to show that

$$(\Omega_{\mathbb{G}_m} L_{\mathbb{A}^1}\mathcal{X})(K) \to (\Omega_{\mathbb{G}_m} L_{\mathrm{Nis}} L^1_{\mathbb{A}}\mathcal{X})(K)$$

is an equivalence. By another induction on the Postnikv tower / descent spectral sequence, we reduce to showing that for $F \in \mathcal{P}_\Sigma(\mathrm{Corr}^{\mathrm{fr}}(k), \mathrm{Ab})$ homotopy invariant, one has

$$H^n_{\mathrm{Nis}}(\mathbb{G}_{mK}, F) = \begin{cases} F(\mathbb{G}_{mK}) & n = 0 \\ 0 & \text{else} \end{cases}.$$

The first case is immediate from Theorem 4.2(1). For the second, note that by Theorem 4.2(3) $a_{\mathrm{Nis}}F$ is strictly homotopy invariant, and hence since $\mathbb{P}^1 \simeq \Sigma\mathbb{G}_m \in \mathcal{S}\mathrm{pc}(k)$ we find that $H^n_{\mathrm{Nis}}(\mathbb{G}_{mK}, F) = H^{n+1}_{\mathrm{Nis}}(\mathbb{P}^1_K, F)$. The result thus follows from the fact that $\mathbb{P}^1$ has Nisnevich cohomological dimension one [MV99, Proposition 3.1.8]. □

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