AFFINE GRASSMANNIANS IN $\mathbb{A}^1$-HOMOTOPY THEORY

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ABSTRACT. Let k be a field. Denote by $\mathcal{S}\mathrm{pc}(k)_*$ the unstable, pointed motivic homotopy category and by $R^{\mathbb{A}^1}\Omega_{\mathbb{G}_m} : \mathcal{S}\mathrm{pc}(k)_* \to \mathcal{S}\mathrm{pc}(k)_*$ the ($\mathbb{A}^1$-derived) $\mathbb{G}_m$-loops functor. For a k-group G, denote by Gr_G the affine Grassmannian of G. If G is isotropic reductive, we provide a canonical motivic equivalence $R^{\mathbb{A}^1}\Omega_{\mathbb{G}_m} G \simeq \mathrm{Gr}_G$. We use this to compute the motive $M(R^{\mathbb{A}^1}\Omega_{\mathbb{G}_m} G) \in \mathcal{DM}(k, \mathbb{Z}[1/e])$.

1. Introduction

This note deals with the subject of $\mathbb{A}^1$-homotopy theory. In other words it deals with the ∞-category $\mathcal{S}\mathrm{pc}(k)$ of motivic spaces over a base field k, together with the canonical functor $\mathrm{Sm}_k \to \mathcal{S}\mathrm{pc}(k)$. Since our results depend crucially on the seminal papers [1, 2, 3], we shall use their definition of $\mathcal{S}\mathrm{pc}(k)$ (which is of course equivalent to the other definitions in the literature): start with the category Sm_k of smooth (separated, finite type) k-schemes, form the universal homotopy theory on Sm_k (i.e. pass to the ∞-category $\mathcal{P}(\mathrm{Sm}_k)$ of space-valued presheaves on k), and then impose the *relations* of Nisnevich descent and contractibility of the affine line $\mathbb{A}^1_k$ (i.e. localise $\mathcal{P}(\mathrm{Sm}_k)$ at an appropriate family of maps).

The ∞-category $\mathcal{S}\mathrm{pc}(k)$ is presentable, so in particular has finite products, and the functor $\mathrm{Sm}_k \to \mathcal{S}\mathrm{pc}(k)$ preserves finite products. Let $* \in \mathcal{S}\mathrm{pc}(k)$ denote the final object (corresponding to the final k-scheme $Spec(k)$); then we can form the pointed unstable motivic homotopy category $\mathcal{S}\mathrm{pc}(k)_* := \mathcal{S}\mathrm{pc}(k)_{*/}$. It carries a symmetric monoidal structure coming from the smash product. Thus, for any $P \in \mathcal{S}\mathrm{pc}(k)_*$ we have the functor $P \wedge \bullet : \mathcal{S}\mathrm{pc}(k)_* \to \mathcal{S}\mathrm{pc}(k)_*$. By abstract nonsense, this functor has a right adjoint $\Omega_P : \mathcal{S}\mathrm{pc}(k)_* \to \mathcal{S}\mathrm{pc}(k)_*$, called the ($\mathbb{A}^1$-derived) P-loops functor.

For us, the most important instance of this is when $P = \mathbb{G}_m$ corresponds to the pointed scheme $\mathbb{G}_m := (\mathbb{A}^1 \setminus 0, 1) \in \mathrm{Sm}_k$. Indeed studying the functor $\Omega_{\mathbb{G}_m}$ is one of the central open problems of unstable motivic homotopy theory, since it is crucial in the passage from unstable to stable motivic homotopy theory. (The functor Ω_{S^1} is similarly important but much better understood.) The main contribution of this note is the following.

Theorem (See Theorem 15)**.** *Let k be a field and G an isotropic reductive k-group. Denote by $\rho(\mathrm{Gr}_G)$ the presheaf on Sm_k represented by the affine Grassmannian of G. Then we have a canonical equivalence*

$$\Omega_{\mathbb{G}_m} G \simeq \rho(\mathrm{Gr}_G)$$

in $\mathcal{S}\mathrm{pc}(k)_*$.

For the somewhat technical notion of isotropic groups, see [2, Definition 3.3.5]. This includes in particular all split groups. For a definition of Gr_G, see [17] or Section 3. For us, the main points are as follows: there exists a pointed presheaf of sets $\mathrm{Gr}_G \in \mathrm{Pre}(\mathrm{Aff}_k)$ (where Aff_k is the category of all affine k-schemes) which is in fact an fpqc sheaf. Moreover, in the category $\mathrm{Pre}(\mathrm{Aff}_k)$, the sheaf Gr_G is a filtered colimit

$$X_1 \to X_2 \to \cdots \to \mathrm{Gr}_G, \tag{1}$$

where each X_i is (the presheaf represented by) a finite type (but in general highly singular) k-scheme.

Classical analog. Our result has the following classical analog. Suppose that $k = \mathbb{C}$. Then the complex points $\mathrm{Gr}_G(\mathbb{C})$ can be given the structure of a topological space, namely the colimit of the spaces $X_i(\mathbb{C})$ (with their strong topology). Then $\mathrm{Gr}_G(\mathbb{C})$ is homeomorphic to the so-called polynomial loop-Grassmannian $\mathrm{Gr}_0^{G(\mathbb{C})}$ of the Lie group $G(\mathbb{C})$ [13, 7.2(i)]. This space is homotopy equivalent to the space of smooth loops $\Omega^{sm}(G(\mathbb{C})')$, where $G(\mathbb{C})'$ is the compact form of $G(\mathbb{C})$ [13, Proposition 8.6.6, Theorem 8.6.2], which itself is well-known to be homotopy equivalent to the usual loop space $\Omega(G(\mathbb{C})')$. Finally since $G(\mathbb{C})' \simeq G(\mathbb{C})$ (by the Iwasawa decomposition) we have $\Omega(G(\mathbb{C})') \simeq \Omega(G(\mathbb{C}))$. Putting everything together, we have found that

$$\mathrm{Gr}_G(\mathbb{C}) \cong \mathrm{Gr}_0^{G(\mathbb{C})} \simeq \Omega^{sm}(G(\mathbb{C})') \simeq \Omega(G(\mathbb{C})') \simeq \Omega(G(\mathbb{C})).$$

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Organisation and further results. In Section 2 we study the interaction of $Sing_*$ and various models of $\Omega_{\mathbb{G}_m}$. Combining this with results of [3], we obtain a preliminary form of our main computation (see Proposition 6): $\Omega_{\mathbb{G}_m} G$ is motivically equivalent to the presheaf

$$X \mapsto G(X[t, t^{-1}])/G(X[t]). \tag{2}$$

In Section 3, we review affine Grassmannians. We make no claims to originality here. The main point is this: Gr_G is usually defined as the fpqc sheafification of the presheaf $X \mapsto G(X(\!(t)\!))/G(X[\![t]\!])$. We show that at least assuming that G is split, this is isomorphic to the Zariski sheafification of (2); see Proposition 9. We also prove that this is an isomorphism on sections over *smooth* affine k-schemes, for any field k, and only assuming that G is isotropic; see Proposition 14. This is enough for our eventual application.

In Section 4, we first deduce the main theorem. This is trivial by now, since Zariski sheafification is a motivic equivalence. After that we explore some consequences. We show in Corollary 19 that if k is perfect, then the $\mathbb{Z}[1/e]$-linear motive of $\rho(\mathrm{Gr}_G) \simeq \Omega_{\mathbb{G}_m} G$ is in fact the filtered colimits of the motives of the singular varieties X_i from (1). Since the geometry of the X_i is well-understood, this allows us in Corollary 20 to determine the motive of $\Omega_{\mathbb{G}_m} G$.

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Notation and conventions. Throughout we use the language of ∞-categories, as set out in [9].

If $\mathcal{C}$ is a small 1-category, we write $\mathcal{P}(\mathcal{C})$ for the ∞-category of presheaves of spaces on $\mathcal{C}$, and $\mathrm{Pre}(\mathcal{C}) = \mathcal{P}(\mathcal{C})_{\le 0}$ for the 1-category of presheaves of sets on $\mathcal{C}$. We denote the ∞-category of spaces by $\mathcal{S}\mathrm{pc}$.

2. $\mathbb{G}_m$-loops of groups

Let $\mathcal{C}$ be an essentially small 1-category with finite products. We write $* \in \mathcal{C}$ for the final object. Throughout we fix $\mathbb{G} \in \mathcal{C}_* := \mathcal{C}_{*/}$. We write $\mathcal{P}(\mathcal{C})$ for the ∞-category of presheaves of spaces on $\mathcal{C}$ and $\mathcal{P}(\mathcal{C})_* := \mathcal{P}(\mathcal{C})_{*/}$ for the pointed version.

We fix a further object $\mathbb{A} \in \mathcal{C}$ together with a map $\mathbb{G} \to \mathbb{A}$. We call $\mathcal{X} \in \mathcal{P}(\mathcal{C})$ $\mathbb{A}$-*invariant* if for all $c \in \mathcal{C}$, the canonical map $\mathcal{X}(c) \to \mathcal{X}(\mathbb{A} \times c)$ is an equivalence.

Example 1. The case we have in mind is, of course, where $\mathcal{C} = \mathrm{Sm}_S$, $\mathbb{G} = \mathbb{G}_m$, and $\mathbb{A} = \mathbb{A}^1$.

Let us recall that the functor $\mathcal{P}(\mathcal{C})_* \to \mathcal{P}(\mathcal{C})_*, \mathcal{X} \mapsto \mathbb{G} \wedge \mathcal{X}$ has a right adjoint $\Omega_{\mathbb{G}} : \mathcal{P}(\mathcal{C})_* \to \mathcal{P}(\mathcal{C})_*$. It is specified in formulas by asserting that the following square is cartesian

$$\begin{array}{ccc} \Omega_{\mathbb{G}}(\mathcal{X})(c) & \longrightarrow & \mathcal{X}(\mathbb{G} \times c) \\ \downarrow & & \downarrow{\scriptstyle i^*} \\ * & \xrightarrow{j_*} & \mathcal{X}(c). \end{array}$$

Here $i : * \to \mathbb{G}$ is the canonical pointing, as is $j : * \to \mathcal{X}$.

For the purpose of this section, by a *group* we shall mean an ∞-group, i.e. a functor $G : \Delta^{\mathrm{op}} \to \mathcal{S}\mathrm{pc}$ such that $G_0 \simeq *$ and $G(S \cup S') \simeq G(S) \times G(S')$ whenever $S \cap S'$ has cardinality 1 [9, Proposition 7.2.2.4]. Denote the ∞-category of ∞-groups by $\mathcal{G}\mathrm{rp}$. Evaluation at [1] induces a forgetful functor $\mathcal{G}\mathrm{rp} \to \mathcal{S}\mathrm{pc}$ which preserves limits and sifted colimits (since limits and colimits in presheaf categories are computed sectionwise [9, Corollary 5.1.2.3], finite products in $\mathcal{S}\mathrm{pc}$ commute with sifted colimits [9, Proposition 5.5.8.6], and sifted categories are weakly contractible [9, Proposition 5.5.8.7]). Given a morphism $H \to G$ of groups, we obtain an action of H on (the underlying space of) G, or in other words a lift of $G : * \to \mathcal{S}\mathrm{pc}$ to a functor $BH \to \mathcal{S}\mathrm{pc}$. We denote by $G/H \in \mathcal{S}\mathrm{pc}$ the colimit of this diagram.

We write $\mathcal{G}\mathrm{rp}(\mathcal{P}(\mathcal{C}))$ for the category of presheaves of groups on $\mathcal{C}$. Suppose that $\mathcal{G} \in \mathcal{G}\mathrm{rp}(\mathcal{P}(\mathcal{C}))$ is a presheaf of groups. Then $\mathcal{G}$ has a canonical pointing, given by the identity section. Thus $\mathcal{G} \in \mathcal{P}(\mathcal{C})_*$, in a canonical way.

Definition 2. We denote by $\Omega^{gr}_{\mathbb{G}}(\mathcal{G}) \in \mathcal{P}(\mathcal{C})$ the presheaf

$$\Omega^{gr}_{\mathbb{G}}(\mathcal{G})(c) = \mathcal{G}(\mathbb{G} \times c)/\mathcal{G}(c),$$

where $\mathcal{G}(c)$ acts on $\mathcal{G}(\mathbb{G} \times c)$ by pulling back along the projection $\mathbb{G} \times c \to c$. We define a further variant

$$\Omega^{gr}_{\mathbb{G},\mathbb{A}}(\mathcal{G})(c) = \mathcal{G}(\mathbb{G} \times c)/\mathcal{G}(\mathbb{A} \times c),$$

where $\mathcal{G}(\mathbb{A} \times c)$ acts on $\mathcal{G}(\mathbb{G} \times c)$ by pullback along $\mathbb{G} \to \mathbb{A}$.

Clearly $\Omega^{gr}_{\mathbb{G}}(\mathcal{G}), \Omega^{gr}_{\mathbb{G},\mathbb{A}}(\mathcal{G})$ are functorial in the presheaf of groups $\mathcal{G}$. Note that unless $\mathcal{G}$ is abelian, $\Omega^{gr}_{\mathbb{G}}(\mathcal{G}), \Omega^{gr}_{\mathbb{G},\mathbb{A}}(\mathcal{G})$ are not a priori a presheaves of groups. Note also that $\mathcal{G}(c) \subset \mathcal{G}(\mathbb{A} \times c)$, and hence there is a canonical map $\Omega^{gr}_{\mathbb{G}}(\mathcal{G}) \to \Omega^{gr}_{\mathbb{G},\mathbb{A}}(\mathcal{G})$.

Proposition 3. *Let $\mathcal{G} \in \mathcal{G}\mathrm{rp}(\mathcal{P}(\mathcal{C}))$ be a presheaf of groups.*

(1) There is a canonical equivalence $\Omega_{\mathbb{G}}(\mathcal{G}) \to \Omega^{gr}_{\mathbb{G}}(\mathcal{G})$.
(2) Suppose that $\mathcal{G}$ is $\mathbb{A}$-invariant. Then the canonical map $\Omega^{gr}_{\mathbb{G}}(\mathcal{G}) \to \Omega^{gr}_{\mathbb{G},\mathbb{A}}(\mathcal{G})$ is a equivalence.

Proof. (1) Let $H \xrightarrow{i} G \xrightarrow{r} H$ be a retraction in $\mathcal{G}\mathrm{rp}$. It suffices to show that there is an equivalence $G/H \simeq fib(r)$, functorial in (G, H, i, r) (apply the construction with $G = \mathcal{G}(c \times \mathbb{G})$ and $H = \mathcal{G}(c)$). Consider the commutative diagram

$$\begin{array}{ccccc} G/H & \longrightarrow & * & \longrightarrow & BH \\ \downarrow & & \downarrow & & \downarrow \\ * & \longrightarrow & X & \longrightarrow & BG \\ & & \downarrow & & \downarrow \\ & & * & \longrightarrow & BH, \end{array}$$

where the bottom right square is defined to be cartesian. The two rectangles are cartesian: this is clear for the vertical one, and for the horizontal one it is essentially the definition of G/H (note that the canonical "inclusion of the fiber" $G/H \to BH$ is null, since it is homotopic to $G/H \to BH \to BG \to BH$). Hence all squares are cartesian, by the pasting law [9, Lemma 4.4.2.1]. Consequently $G/H \simeq \Omega X$. Recall that $\mathcal{G}\mathrm{rp}$ is equivalent to the category $\mathcal{S}\mathrm{pc}^{\geq 1}_*$ of pointed connected spaces [9, Lemma 7.2.2.11], and that the functor B is given by $\mathcal{G}\mathrm{rp} \simeq \mathcal{S}\mathrm{pc}^{\geq 1}_* \to \mathcal{S}\mathrm{pc}$. It follows that B preserves pullbacks if one of the maps is a surjection on π_0 and hence $X \simeq Bfib(r)$. Thus $G/H \simeq \Omega B fib(r) \simeq fib(r)$, as was to be shown.

(2) Since $\mathcal{G}(c) \simeq \mathcal{G}(\mathbb{A} \times c)$ by assumption, $\Omega^{gr}_{\mathbb{G}}(\mathcal{G})(c) = \mathcal{G}(\mathbb{G} \times c)/\mathcal{G}(c)$ and $\Omega^{gr}_{\mathbb{G},\mathbb{A}}(\mathcal{G})(c) = \mathcal{G}(\mathbb{G} \times c)/\mathcal{G}(\mathbb{A} \times c)$ are colimits of equivalent diagrams, and hence equivalent. □

To go further, we need to assume that $\mathbb{A}$ is given the structure of a *representable interval object* [1, Definition 4.1.1]. In this case there is a functor

$$\mathrm{Sing} : \mathcal{P}(\mathcal{C})_* \to \mathcal{P}(\mathcal{C})_*, \mathcal{X} \mapsto |\operatorname{Sing}_*(\mathcal{X})|,$$

where $\operatorname{Sing}_*(\mathcal{X}) \in \mathrm{Fun}(\Delta^{\mathrm{op}}, \mathcal{P}(\mathcal{C})_*)$ is given by $[n] \mapsto \mathcal{X}(\mathbb{A}^n \times -)$ and $|\ldots|$ denotes geometric realization. The functor Sing is a functorial "$\mathbb{A}$-localization"; in particular it produces $\mathbb{A}$-invariant objects. All of these properties are mentioned in [1], right after Definition 4.1.4. Moreover since $\mathcal{G}\mathrm{rp} \to \mathcal{S}\mathrm{pc}$ preserves sifted colimits, Sing maps presheaves of groups to presheaves of groups.

Lemma 4. *Let $\mathcal{G} \in \mathcal{G}\mathrm{rp}(\mathcal{P}(\mathcal{C}))$. Then $\Omega_{\mathbb{G}}, \Omega^{gr}_{\mathbb{G}}$ and $\Omega^{gr}_{\mathbb{G},\mathbb{A}}$ commute with* Sing *when applied to $\mathcal{G}$. In other words, there are canonical equivalences*

$$\begin{aligned} \Omega_{\mathbb{G}} \operatorname{Sing} \mathcal{G} &\simeq \operatorname{Sing} \Omega_{\mathbb{G}} \mathcal{G}, \\ \Omega^{gr}_{\mathbb{G}} \operatorname{Sing} \mathcal{G} &\simeq \operatorname{Sing} \Omega^{gr}_{\mathbb{G}} \mathcal{G}, \text{ and} \\ \Omega^{gr}_{\mathbb{G},\mathbb{A}} \operatorname{Sing} \mathcal{G} &\simeq \operatorname{Sing} \Omega^{gr}_{\mathbb{G},\mathbb{A}} \mathcal{G}. \end{aligned}$$

Proof. Since colimits commute, the claims about $\Omega^{gr}_{\mathbb{G}}$ and $\Omega^{gr}_{\mathbb{G},\mathbb{A}}$ are clear. Since $\Omega_{\mathbb{G}} \simeq \Omega^{gr}_{\mathbb{G}}$ when applied to presheaves of groups, by Proposition 3(1), the claim about $\Omega_{\mathbb{G}}$ also follows. □

Corollary 5. *Let $\mathcal{G} \in \mathcal{G}\mathrm{rp}(\mathcal{P}(\mathcal{C}))$. Then $\Omega_{\mathbb{G}} \operatorname{Sing} \mathcal{G} \simeq \operatorname{Sing} \Omega_{\mathbb{G}} \mathcal{G} \simeq \operatorname{Sing} \Omega^{gr}_{\mathbb{G}} \mathcal{G} \simeq \operatorname{Sing} \Omega^{gr}_{\mathbb{G},\mathbb{A}} \mathcal{G}$.*

Proof. We have

$$\Omega_{\mathbb{G}} \operatorname{Sing} \mathcal{G} \simeq \operatorname{Sing} \Omega_{\mathbb{G}} \mathcal{G} \simeq \operatorname{Sing} \Omega^{gr}_{\mathbb{G}} \mathcal{G} \simeq \Omega^{gr}_{\mathbb{G}} \operatorname{Sing} \mathcal{G} \simeq \Omega^{gr}_{\mathbb{G},\mathbb{A}} \operatorname{Sing} \mathcal{G} \simeq \operatorname{Sing} \Omega^{gr}_{\mathbb{G},\mathbb{A}} \mathcal{G},$$

applying Lemma 4 and Proposition 3 alternatingly. □

Specialisation to $\mathbb{A}^1$-homotopy theory. We now consider the situation where $\mathcal{C} = \mathrm{Sm}_S^{\mathrm{aff}}$, $\mathbb{G} = \mathbb{G}_m, \mathbb{A} = \mathbb{A}^1$. Here S is a Noetherian scheme of finite Krull dimension (in all our applications it will be the spectrum of a field), and $\mathrm{Sm}_S^{\mathrm{aff}}$ denotes the category of smooth, finite type, (relative) affine S-schemes.

We write $L_{\mathrm{mot}}\mathcal{P}(\mathrm{Sm}_S^{\mathrm{aff}})$ for the motivic localization of the ∞-category $\mathcal{P}(\mathrm{Sm}_S^{\mathrm{aff}})$; in other words the localization inverting $\mathbb{A}^1$-homotopy equivalences and the distinguished Nisnevich squares (equivalently, the Nisnevich-local weak equivalences [10, Lemma 3.1.18]). It is well-known that $L_{\mathrm{mot}}\mathcal{P}(\mathrm{Sm}_S^{\mathrm{aff}})$ is equivalent to $L_{\mathrm{mot}}\mathcal{P}(\mathrm{Sm}_S)$, the usual model for the pointed, unstable motivic homotopy category. Indeed already $L_{\mathrm{Nis}}\mathcal{P}(\mathrm{Sm}_S^{\mathrm{aff}}) \simeq L_{\mathrm{Nis}}\mathcal{P}(\mathrm{Sm}_S)$, because Sm_S and $\mathrm{Sm}_S^{\mathrm{aff}}$ define the same site. We write $L_{\mathrm{mot}} : \mathcal{P}(\mathrm{Sm}_S^{\mathrm{aff}}) \to \mathcal{P}(\mathrm{Sm}_S^{\mathrm{aff}})$ for the localization functor.

We can now state the main result of this section.

Proposition 6. *Let k be a field and G an isotropic reductive k-group. Then*

$$\Omega_{\mathbb{G}_m} L_{\mathrm{mot}} G \simeq \mathrm{Sing}\,\Omega_{\mathbb{G}_m} G \simeq \mathrm{Sing}\,\Omega_{\mathbb{G}_m}^{gr} G \simeq \mathrm{Sing}\,\Omega_{\mathbb{G}_m,\mathbb{A}^1}^{gr} G.$$

Proof. The main point is that under our assumptions, $L_{\mathrm{mot}} G \simeq \mathrm{Sing}\, G$ [3, Theorem 2.6] [2, Definiton 2.1.1]. The result is now just a restatement of Corollary 5. □

Remark 7. The presheaves of spaces $\Omega_{\mathbb{G}_m}^{gr} G$ and $\Omega_{\mathbb{G}_m,\mathbb{A}^1}^{gr} G$ are discrete, or in other words the relevant quotients may be computed in the category of sets. This is because an (ordinary) quotient by a free group action (such as a subgroup acting on the larger group) is a homotopy colimit, and the maps $G(X) \to G(X \times \mathbb{G}_m)$ and $G(X \times \mathbb{A}^1) \to G(X \times \mathbb{G}_m)$ are injections for every $X \in \mathrm{Sch}_k$ (the latter since G is affine and hence separated, by assumption [2, Definition 3.1.1]).

3. Affine Grassmannians

We review some basic results about affine Grassmannians. Surely they are all well-known to workers in the field (i.e., not the author). Our main reference is [17]. Throughout, we fix a field k and write Aff_k for the category of all affine k-schemes (not necessarily of finite type, not necessarily smooth). We extensively work in the category $\mathrm{Pre}(\mathrm{Aff}_k)$ of presheaves of sets on affine schemes; as is well-known we have the Yoneda embedding $\mathrm{Sch}_k \to \mathrm{Pre}(\mathrm{Aff}_k)$. On $\mathrm{Pre}(\mathrm{Aff}_k)$ we have many topologies, the most relevant for us are the *fpqc* topology [14, Tag 03NV] and the Zariski topology; we denote the relevant sheafification functors by a_{fpqc} (which may not always exist!) and a_{Zar}. For objects $\mathcal{F} \in \mathrm{Pre}(\mathrm{Aff}_k)$ and A any k-algebra, we put $\mathcal{F}(A) := \mathcal{F}(Spec(A))$.

Definition 8. Let $\mathcal{X} \in \mathrm{Pre}(\mathrm{Aff}_k)$ be a presheaf. We have the presheaves $L^+\mathcal{X}, L\mathcal{X} \in \mathrm{Pre}(\mathrm{Aff}_k)$ defined by

$$L^+\mathcal{X}(A) = \mathcal{X}(A[\![t]\!])$$

and

$$L\mathcal{X}(A) = \mathcal{X}(A(\!(t)\!)).$$

Note that there is a canonical morphism $L^+\mathcal{X} \to L\mathcal{X}$ induced by $A[\![t]\!] \to A(\!(t)\!)$.

Let $\mathcal{G} \in \mathrm{Pre}(\mathrm{Aff}_k)$ be a presheaf of groups. Then $L^+\mathcal{G}, L\mathcal{G}$ are presheaves of groups and we define the *affine Grassmannian* as

$$\mathrm{Gr}_{\mathcal{G}} = a_{\mathrm{fpqc}} L\mathcal{G}/L^+\mathcal{G}.$$

We note right away that at least if $\mathcal{G}$ is represented by a group scheme, then $\mathrm{Gr}_{\mathcal{G}} = a_{\mathrm{fpqc}} L\mathcal{G}/L^+\mathcal{G}$ exists and is given by $a_{\mathrm{\acute{e}t}} L\mathcal{G}/L^+\mathcal{G}$ [17, Proposition 1.3.6, Lemma 1.3.7]. Let us further put $L_0\mathcal{G}(A) = \mathcal{G}(A[t, t^{-1}])$ and $L_0^+\mathcal{G}(A) = \mathcal{G}(A[t])$. Then we have a commutative square

$$\begin{array}{ccc} L_0\mathcal{G} & \longrightarrow & L\mathcal{G} \\ \uparrow & & \uparrow \\ L_0^+\mathcal{G} & \longrightarrow & L^+\mathcal{G}. \end{array} \tag{3}$$

The main result of this section is the following. See also Proposition 14 at the end of this section for a related and sometimes stronger result.

Proposition 9. *Let G be a split reductive group over a field k. Then the canonical map*

$$a_{\mathrm{Zar}} L_0 G / L_0^+ G \to \mathrm{Gr}_G$$

induced by (3) *is an isomorphism (of objects in* $\mathrm{Pre}(\mathrm{Aff}_k)$*).*

Before giving the proof, we need some background material. If τ is a topology, we call a morphism of presheaves $f : \mathcal{X} \to \mathcal{Y}$ a τ-epimorphism if $a_\tau f$ is an epimorphism in the topos of τ-sheaves.

Definition 10. Let $\mathcal{G} \in \mathrm{Pre}(\mathrm{Aff}_k)$ be a presheaf of groups acting on $\mathcal{X} \in \mathrm{Pre}(\mathrm{Aff}_k)$. Suppose given a $\mathcal{G}$-equivariant map $f : \mathcal{X} \to \mathcal{Y}$, where $\mathcal{Y} \in \mathrm{Pre}(\mathrm{Aff}_k)$ has the trivial $\mathcal{G}$-action. Let τ be a topology on Aff_k. We call f a τ-locally trivial $\mathcal{G}$-torsor if:

(1) $\mathcal{G}, \mathcal{X}, \mathcal{Y}$ are τ-sheaves,
(2) f is a τ-epimorphism, and
(3) the canonical map $\mathcal{G} \times \mathcal{X} \to \mathcal{X} \times_{\mathcal{Y}} \mathcal{X}, (g, x) \mapsto (x, gx)$ is an isomorphism.

Let us note that condition (1) implies that $\mathcal{G} \times \mathcal{X}$ and $\mathcal{X} \times_{\mathcal{Y}} \mathcal{X}$ are τ-sheaves, so condition (3) is τ-local. We call a $\mathcal{G}$-torsor *trivial* if there is a $\mathcal{G}$-equivariant isomorphism $\mathcal{X} \cong \mathcal{G} \times \mathcal{Y}$.

Lemma 11. *Suppose that $\mathcal{G}$ is a presheaf of groups acting on $\mathcal{X}$, and $f : \mathcal{X} \to \mathcal{Y}$ is a $\mathcal{G}$-equivariant map, where $\mathcal{G}$ acts trivially on $\mathcal{Y}$. Suppose that $\mathcal{G}, \mathcal{X}, \mathcal{Y}$ are τ-sheaves. The following are equivalent.*

(1) f is a τ-locally trivial $\mathcal{G}$-torsor.
(2) For every affine scheme S and every morphism $S \to \mathcal{Y}$, there exists a τ-cover $\{S_i \to S\}_i$ such that $\mathcal{X}_{S_i}$ is a trivial $\mathcal{G}$-torsor (for every i).
(3) There exists a τ-epimorphism $\mathcal{U} \to \mathcal{Y}$ such that $\mathcal{X}_{\mathcal{U}} \to \mathcal{X}$ is a trivial $\mathcal{G}$-torsor.

Proof. We will work in the topos of τ-sheaves, so suppress any mention of τ-sheafification, and also say "epimorphism" instead of "τ-epimorphisms", and so on.

(1) $\Rightarrow$ (2): Let $S \to \mathcal{Y}$ be any map. Since epimorphisms in a topos are stable under base change (e.g. by universality of colimits), $\alpha : \mathcal{X}_S \to S$ is also a $\mathcal{G}$-torsor, and in particular an epimorphism. There exists then a cover $\{S_i \to S\}_i$ over which α has a section, being an epimorphism. In other words, $\mathcal{X}_{S_i} \to S'$ is trivial, as required.

(2) $\Rightarrow$ (3): Taking the coproduct $\coprod_{S \to \mathcal{Y}} \coprod_i S_i \to \mathcal{Y}$ over a sufficiently large collection of affine schemes S mapping to $\mathcal{Y}$, we obtain a trivializing epimorphism as required.

(3) $\Rightarrow$ (1): We need to prove that $\mathcal{X} \to \mathcal{Y}$ is an epimorphism and that $\mathcal{G} \times \mathcal{X} \to \mathcal{X} \times_{\mathcal{Y}} \mathcal{X}$ is an isomorphism. Both statements may be checked after pullback along the (effective) epimorphism $\mathcal{U} \to \mathcal{Y}$. We may thus assume that $\mathcal{X} \to \mathcal{Y}$ is trivial, in which case the result is clear. □

Lemma 12. *Let $\mathcal{X} \to \mathcal{Y}$ be a τ-locally trivial $\mathcal{G}$-torsor. Then $\mathcal{Y} \cong a_\tau \mathcal{X}/\mathcal{G}$.*

Proof. We again work in the topos of τ-sheaves. By definition $\mathcal{X} \to \mathcal{Y}$ is an epimorphism. Since every epimorphism in a topos is effective [14, Tag 086K], we have a coequaliser $\mathcal{X} \times_{\mathcal{Y}} \mathcal{X} \rightrightarrows \mathcal{X} \to \mathcal{Y}$ in τ-sheaves. By condition (3) of Definition 10, this is the action coequalizer. The result follows. □

We now supply a geometric proof of Proposition 9. It only works if k is infinite. We supply a more abstract proof which works in general further down below.

Proof of Proposition 9, assuming k infinite. By Lemma 12, it suffices to prove that $L_0 G \to \mathrm{Gr}_G$ is a Zariski-locally trivial $L_0^+ G$-torsor. All presheaves involved are fpqc-, and hence Zariski-sheaves.

We shall make use of results from [17, Section 2]. There k is assumed to be algebraically closed. This will not matter in each case we cite this reference, because the property we are checking will be fpqc local.

We introduce some additional notation. We denote by $L^- G$ the presheaf $A \mapsto G(A[t^{-1}])$. We have $L^- G \cong L_0^+ G$, but the canonical embedding $L^- G \to L_0 G$ is different. Evaluation at $t^{-1} = 0$ induces $L^- G \to G$, and we let $L^{<0} G = ker(L^- G \to G)$. I claim that the following square is a pullback, where i is the multipication map

$$\begin{array}{ccc} L^{<0} G \times L_0^+ G & \xrightarrow{i} & L_0 G \\ {\scriptstyle pr_1}\downarrow & & \downarrow \\ L^{<0} G & \xrightarrow{j} & \mathrm{Gr}_G. \end{array}$$

In order to see this, we note first that it follows from [7, Lemma 3.1] that i is a monomorphism. Let $\mathcal{F} = L^{<0} G \times_{\mathrm{Gr}_G} L_0 G$. Since i is a mono so is the canonical map $\alpha : L^{<0} G \times L_0^+ G \to \mathcal{F}$. Let $(x, y) \in \mathcal{F}(A)$, corresponding to $x \in L^{<0} G(A)$ and $y \in L_0 G(A)$ with the same image in $\mathrm{Gr}_G(A)$. In other words, fpqc-locally on A we can find $z \in L_0^+ G(A)$ with $y = xz$. Thus α is fpqc-locally an epimorphism. It is thus an fpqc-local isomorphism of fpqc-sheaves, and hence an isomorphism (of presheaves).

Now let $A \in L_0G(k)$. We obtain a map $j_A : L^{<0}G \to \mathrm{Gr}_G, x \mapsto A \cdot j(x)$. Define similarly $i_A : L^{<0}G \times L_0^+G \to L_0G, (x, y) \mapsto Axy$. Since A is invertible, the following square is also a pullback

$$\begin{array}{ccc} L^{<0}G \times L_0^+G & \xrightarrow{i_A} & L_0G \\ {\scriptstyle pr_1}\downarrow & & \downarrow \\ L^{<0}G & \xrightarrow{j_A} & \mathrm{Gr}_G. \end{array}$$

By Lemma 11, it is thus enough to show that $j' = \coprod_{A \in L_0G(k)} j_A$ is a Zariski-epimorphism. Note first that j_A is a morphism of ind-schemes [17, Theorem 1.2.2], and an open immersion [7, Lemma 3.1]. Consequently each j_A identifies an open ind-subscheme. In order to check that j' is a Zariski-epimorphism, it suffices to check that the j_A form a covering. Let $\bar{k}$ denote an algebraic closure of k. Since Gr_G is of ind-finite type, it suffices to check this on $\bar{k}$-points. The result thus follows from Lemma 13 below. □

In the above proof, we used the following result, which is trivial if $k = \bar{k}$. The general case is probably well-known to experts. A proof was kindly communicated by Timo Richarz.

Lemma 13. *Let k be an infinite field, $\bar{k}$ an algebraic closure, and G a split reductive group over k. Then $\mathrm{Gr}_G(\bar{k})$ is covered by the translates $AL^{<0}G(\bar{k}) \subset \mathrm{Gr}_G(\bar{k})$, for $A \in L_0G(k)$.*

Proof. We shall make use of the *Bruhat decomposition* of Gr_G. Namely, there exists a set X, together with for each $\mu \in X$ an element $t^\mu \in L_0G(k)$ and a k-scheme $U_\mu \subset L_0G$ such that

(1) The canonical map $U_\mu \to \mathrm{Gr}_G, A \mapsto At^\mu \cdot e$ is a locally closed embedding. Denote the image by Y_μ.
(2) There is an isomorphism $U_\mu \cong \mathbb{A}^{l(\mu)}$ for some non-negative integer $l(\mu)$.
(3) The schemes $Y_\mu \to \mathrm{Gr}_G$ form a locally closed cover.

We do not know a good reference for the statement in this generality, but see for example [13, Theorem 8.6.3].

It is clear that $L_0G \to \mathrm{Gr}_G$ is trivial over Y_μ. We deduce that (1) $L_0G \to \mathrm{Gr}_G$ is surjective on k-points. Put $\bar{G} = G_{\bar{k}}$. We claim that (2) any non-empty open $L_0^+\bar{G}$-orbit in $L_0\bar{G}$ contains (the image of) a k-point (of L_0G). Using surjectivity on k-points, for this it suffices to prove that any non-empty open $U \subset \mathrm{Gr}_{\bar{G}}$ contains a k-rational point. Being non-empty, U meets $\bar{Y}_\mu := (Y_\mu)_{\bar{k}}$ for some μ. Then $\bar{V} := \bar{Y}_\mu \cap U$ is a non-empty open subset of $\bar{Y}_\mu \cong \mathbb{A}^n_{\bar{k}}$ for some n. Its image V in $\mathbb{A}^n_k$ is open [14, Tag 0383] and non-empty. Since k is infinite, V has a rational point*. This establishes the claim.

Finally let $\bar{A} \in \mathrm{Gr}_G(\bar{k})$. By surjectivity on $\bar{k}$-points (1), we find $A \in L_0G(\bar{k})$ mapping to $\bar{A}$. Consider the $L^-\bar{G}$-orbit $O = AL_0^+\bar{G}L^-\bar{G} \subset L_0\bar{G}$. I claim that O contains a k-point. The automorphism $\mathrm{rev} : L_0G \to L_0G$ induced by $t \mapsto t^{-1}$ interchanges L^- and L_0^+, and hence converts L^- orbits into L_0^+-orbits. Since it is defined over k it preserves k-points. It is hence enough to show $\mathrm{rev}(O)$ has a k-point, and by the claim (2) it is enough to show that $\mathrm{rev}(O)$ is open. But $\mathrm{rev}(O) = \mathrm{rev}(A)L^-\bar{G}L_0^+\bar{G}$ which is open, being the preimage of $\mathrm{rev}(A)L^{<0}\bar{G} \subset \mathrm{Gr}_{\bar{G}}$.

We thus find $B \in L_0^+G(\bar{k}), C \in L^-G(\bar{k})$ such that $ABC \in L_0G(k)$. Then

$$\begin{aligned} \bar{A} = A \cdot e &= (ABC)C^{-1}B^{-1} \cdot e = (ABC)C^{-1} \cdot e \\ &\in (ABC)L^-G(\bar{k}) \cdot e = (ABC)L^{<0}G(\bar{k}) \cdot e \subset \mathrm{Gr}_G(\bar{k}). \end{aligned}$$

This was to be shown. □

Now we come to an alternative proof of Proposition 9, using a recent result of Fedorov [5]. This proof does not require k to be infinite, or a field. It was also kindly communicated by Timo Richarz.

Proof of Proposition 9, general case. It follows from the Beauville-Laszlo gluing lemma [4] that $\mathrm{Gr}_G \cong a_{fpqc}L_0G/L_0^+G \cong T$, where T is the functor sending $Spec(A)$ to the set of isomorphism classes of tuples $(\mathcal{F}, \alpha)$ with $\mathcal{F}$ a G-torsor on $\mathbb{A}^1_A$ and α a trivialization of $\mathcal{F}$ over $\mathbb{A}^1_A \setminus \{0\}$. The map $L_0G \to T$ sends $M \in L_0G(A)$ to the pair $(\mathcal{F}^0, \alpha_M)$, where $\mathcal{F}^0$ is the trivial G-torsor and α_M is the trivialization induced by M. By Lemmas 11 and 12, what we need to show is that this map $L_0G \to T$ admits sections Zariski-locally on T. In other words if $Spec(A) \in \mathrm{Aff}_k$ and $(\mathcal{F}, \alpha) \in T(A)$, then the G-torsor $\mathcal{F}$ over $\mathbb{A}^1_A$ is Zariski-locally on A trivial.

*This result is widely known and easy to prove, yet we could not locate a reference. A proof is recorded on MathOverflow at [12].

If A is Noetherian local, this is [5, Theorem 2]. We need to extend this to more general A, so let $Spec(A) \in \mathrm{Aff}_k$ and $(\mathcal{F}, \alpha) \in T(A)$ be arbitrary. We may write A as a filtering colimit of Noetherian rings A_i. Since Gr_G is of ind-finite type, we find $(\mathcal{F}', \alpha') \in T(A_i)$ for some i inducing $(\mathcal{F}, \alpha)$. Thus we may assume that A is Noetherian. Fedorov's result assures us that $\mathcal{F}$ is trivial over any local ring of A. By quasi-compactness, given $P \in Spec(A)$ and a trivialization of $\mathcal{F}$ over $\mathbb{A}^1_{A_P}$, there exists $f \in A \setminus P$ such that the trivialization extends over $\mathbb{A}^1_{A_f}$. Thus $\mathcal{F}$ is Zariski-locally trivial, as was to be shown. □

We can also prove the following related result, tailored to our narrower applications.

Proposition 14. *Let G be an isotropic reductive group over a field k. Then the canonical map*

$$a_{\mathrm{Zar}} L_0 G / L_0^+ G \to \mathrm{Gr}_G$$

induced by (3) *is an isomorphism on sections over smooth affine varieties.*

Proof. By arguing as in the alternative proof of Proposition 9, what we need to show is the following: if X is a smooth affine variety and $\mathcal{F}$ is a G-torsor on $\mathbb{A}^1_X$ which is trivial over $\mathbb{A}^1_X \setminus \{0\}$, then $\mathcal{F}$ is Zariski-locally on X trivial. By definition $\mathcal{F}$ is generically trivial, and hence by the resolution of the Grothendieck-Serre conjecture over fields [6, 11], $\mathcal{F}$ is Zariski-locally trivial (on $\mathbb{A}^1_X$). By homotopy invariance for G-torsors over smooth affine schemes [3, Theorem 2.4], we find that $\mathcal{F} \cong (\mathbb{A}^1_X \to X)^* \mathcal{G}$, for some Nisnevich-locally trivial G-torsor $\mathcal{G}$ on X. Now $\mathcal{G}$ is generically trivial, so by Grothendieck-Serre again $\mathcal{G}$ is Zariski-locally trivial. This concludes the proof. □

4. Main result

We now come to our main result. Let $\mathcal{S}\mathrm{pc}(k)_*$ denote the ∞-category of pointed motivic spaces. As usual we have a canonical functor $(\mathrm{Sm}_k)_* \to \mathcal{S}\mathrm{pc}(k)_*$. We also have the functor $\rho : \mathrm{Pre}(\mathrm{Aff}_k)_* \to \mathcal{S}\mathrm{pc}(k)_*$. It is obtained as the composite

$$\mathrm{Pre}(\mathrm{Aff}_k)_* \xrightarrow{j^*} \mathrm{Pre}(\mathrm{Sm}_k^{\mathrm{aff}})_* \xrightarrow{L} \mathcal{S}\mathrm{pc}(k)_*,$$

where the j^* is restriction along $j : \mathrm{Sm}_k^{\mathrm{aff}} \to \mathrm{Aff}_k$ and L is the motivic localization functor. Recall also the $\mathbb{G}_m$-loops functor $R^{\mathbb{A}^1}\Omega_{\mathbb{G}_m} : \mathcal{S}\mathrm{pc}(k)_* \to \mathcal{S}\mathrm{pc}(k)_*$.

Theorem 15. *Let k be a field and G an isotropic reductive k-group. Then we have a canonical equivalence*

$$R^{\mathbb{A}^1}\Omega_{\mathbb{G}_m} G \simeq \rho(\mathrm{Gr}_G)$$

in $\mathcal{S}\mathrm{pc}(k)_*$. Here Gr_G is pointed by the image of the identity element in G.

Proof. By Proposition 6, we have $R^{\mathbb{A}^1}\Omega_{\mathbb{G}_m} G = \Omega_{\mathbb{G}_m} L_{mot} G \simeq \Omega^{gr}_{\mathbb{G}_m, \mathbb{A}^1} G$, a weak equivalence in $\mathcal{S}\mathrm{pc}(k)_*$. By Remark 7, in the notation of Section 3, we have $\Omega^{gr}_{\mathbb{G}_m, \mathbb{A}^1} G = j^*(L_0 G / L_0^+ G)$. For $F \in \mathrm{Pre}(\mathrm{Sm}_S^{\mathrm{aff}})_*$ the map $F \to a_{\mathrm{Zar}} F$ is a motivic equivalence, i.e. becomes an equivalence in $\mathcal{S}\mathrm{pc}(k)_*$. Since j^* commutes with a_{Zar}, the result now follows from Proposition 14. □

Example 16. Group schemes G satisfying the assumptions of Theorem 15 are, among many others, $\mathrm{GL}_n, \mathrm{SL}_n, \mathrm{Sp}_n$.

Motives of singular varieties. The presheaves Gr_G are well-understood: they are filtered colimits of projective varieties. Unfortunately, these projective varieties are highly singular. Thus we need to incorporate motives of singular varieties in order to make the best use of Theorem 15.

Let Ft_k denote the category of finite type k-schemes, and suppose that k has exponential characteristic e (i.e. $e = 1$ if $char(k) = 0$ and $e = p$ if $char(k) = p > 0$). Recall the ∞-category $\mathcal{DM}(k, \mathbb{Z}[1/e])$ of $\mathbb{Z}[1/e]$-linear motives [15] and the functor $M : \mathcal{S}\mathrm{pc}(k)_* \to \mathcal{DM}(k, \mathbb{Z}) \to \mathcal{DM}(k, \mathbb{Z}[1/e])$ sending a pointed motivic space to its motive. There also is a more complicated functor

$$\underline{M} : \mathrm{Pre}(\mathrm{Ft}_k)_* \to \mathcal{DM}(k, \mathbb{Z}[1/e]);$$

we recall its definition below in the proof of Proposition 17. For $X \in (\mathrm{Sm}_k)_*$ we have $MX \simeq \underline{M}X$, where on the right hand side we view X as an element of $(\mathrm{Ft}_k)_* \subset \mathrm{Pre}(\mathrm{Ft}_k)_*$. In other words, the functor $\underline{M}$ allows us to make sense the motive of (among other things) singular varieties.

Denote by $e^* : \mathrm{Pre}(\mathrm{Ft}_k)_* \to \mathrm{Pre}(\mathrm{Sm}_k^{\mathrm{aff}})_*$ the functor of restriction along the canonical inclusion $\mathrm{Sm}_k^{\mathrm{aff}} \to \mathrm{Ft}_k$.

Proposition 17. *Let k be a perfect field and $\mathcal{X} \in \mathrm{Pre}(\mathrm{Ft}_k)_*$. Then $Me^*\mathcal{X} \simeq \underline{M}\mathcal{X}$.*

Proof. For a small (1-)category $\mathcal{C}$, denote by $\mathcal{P}(\mathcal{C})$ the ∞-category of (space-valued) presheaves on $\mathcal{C}$. If $f : \mathcal{C} \to \mathcal{D}$ is a functor, we get an adjunction $f : \mathcal{P}(\mathcal{C}) \leftrightarrows \mathcal{P}(\mathcal{D}) : f^*$. We have a full inclusion $\mathrm{Pre}(\mathcal{C}) \subset \mathcal{P}(\mathcal{C})$ and similarly for $\mathcal{D}$, and the following diagram commutes:

$$\begin{array}{ccc} \mathrm{Pre}(\mathcal{C}) & \longrightarrow & \mathcal{P}(\mathcal{C}) \\ {\scriptstyle f^*}\uparrow & & {\scriptstyle f^*}\uparrow \\ \mathrm{Pre}(\mathcal{D}) & \longrightarrow & \mathcal{P}(\mathcal{D}). \end{array}$$

The functor $\underline{M}$ is constructed via the following commutative diagram

$$\begin{array}{ccc} \mathcal{P}(\mathrm{Ft}_k)_* & \xrightarrow{\mu} & \underline{\mathcal{DM}}(k, \mathbb{Z}[1/e]) \\ {\scriptstyle e}\uparrow & & {\scriptstyle e^M}\uparrow \\ \mathcal{P}(\mathrm{Sm}_k^{\mathrm{aff}})_* & \xrightarrow{M} & \mathcal{DM}(k, \mathbb{Z}[1/e]). \end{array}$$

The category $\underline{\mathcal{DM}}(k, \mathbb{Z}[1/e])$ can be defined as the T-stabilisation of $L_{cdh,\mathbb{A}^1}\mathcal{P}_\Sigma(Cor(k, \mathbb{Z}[1/e]))$, where $Cor(k, \mathbb{Z}[1/e])$ is the category of finite correspondences with $\mathbb{Z}[1/e]$-coefficients (see e.g. [8, Definition 2.3.1] and $\mathcal{P}_\Sigma$ denotes the nonabelian derived category [9, Section 5.5.8]. The functor e^M is the stabilisation of the derived left Kan extension functor $e : \mathcal{P}_\Sigma(SmCor(k)) \to \mathcal{P}_\Sigma(Cor(k))$. The important result is that e^M is an equivalence [8, Corollary 4.0.14]; one puts $\underline{M} = (e^M)^{-1} \circ \mu$.

In order to prove our result, it is thus enough to show that the co-unit map $\eta : ee^*\mathcal{X} \to \mathcal{X}$ is inverted by the functor μ. For this it suffices to show that the image $\mu_l(\eta) \in \underline{\mathcal{DM}}(k, \mathbb{Z}_{(l)})$ of $\mu(\eta)$ is an equivalence for all primes $l \neq p$. The functor μ_l inverts local equivalences for the so-called ldh-topology [8, Corollary 4.0.14] and all finite type k-schemes are ldh-locally smooth [8, Corollary 2.1.15]. It is hence enough to show that $e^*(\eta) : e^*ee^*\mathcal{X} \to e^*\mathcal{X}$ is an equivalence. This follows from the fact that $e^*e \simeq \mathrm{id}_{\mathcal{P}(\mathrm{Sm}_k^{\mathrm{aff}})_*}$, which itself is a consequence of fully faithfulness of $e : \mathrm{Sm}_k^{\mathrm{aff}} \to \mathrm{Ft}_k$. □

Remark 18. If k has characteristic 0, then using [16] for $\mathcal{X} \in \mathrm{Pre}(\mathrm{Ft}_k)_*$ one may define the S^1-stable homotopy type $\underline{\Sigma}_s^\infty \mathcal{X} \in \mathcal{SH}^{S^1}(k)$. Essentially the same proof as above shows that $\Sigma_s^\infty e^*\mathcal{X} \simeq \underline{\Sigma}_s^\infty \mathcal{X} \in \mathcal{SH}^{S^1}(k)$. In positive characteristic, there does not seem to be an equally accessible S^1-stable homotopy type of singular varieties.

Corollary 19. *Let $X_1 \to X_2 \to \cdots \in (\mathrm{Ft}_k)_*$ be a directed system of pointed finite type k-schemes. View each X_i as an element of $\mathrm{Pre}(\mathrm{Aff}_k)_*$ and put $\mathcal{X} = \mathrm{colim}_i X_i \in \mathrm{Pre}(\mathrm{Aff}_k)_*$. Then we have $M\rho(\mathcal{X}) \simeq \mathrm{colim}_i \underline{M}X_i$.*

We note that a filtered colimit of fpqc-sheaves (computed in $\mathrm{Pre}(\mathrm{Aff}_k)$) is an fpqc-sheaf [14, Tags 0738 (4) and 022E], and so the colimit in the corollary can be computed in the category of fpqc-sheaves.

Proof. Let $\mathcal{X}' \in \mathrm{Pre}(\mathrm{Ft}_k)_*$ be the colimit viewed as a presheaf on finite type schemes. Then $e^*(\mathcal{X}') = j^*(\mathcal{X})$ and the result follows from Proposition 17, using that all functors in sight commute with filtered colimits. □

Corollary 20. *Let G be an isotropic reductive group over a perfect field k of exponential characteristic e. Then we have*

$$M(\rho(\mathrm{Gr}_G)) \simeq \bigoplus_{\mu \in X(G)} \mathbb{Z}[1/e](l(\mu))[2l(\mu)] \in \mathcal{DM}(k, \mathbb{Z}[1/e]).$$

Here $X(G)$ is the set of cocharacters of G, and $l(\mu)$ is the (non-negative) integer from the proof of Lemma 13.

Proof. The Bruhat decomposition provides a filtration of Gr_G by closed subschemes $\emptyset = X_{-1} \subset X_0 \subset X_1 \subset \cdots \subset \mathrm{Gr}_G$. Here $X_i = \cup_{l(\mu)\le i} Y_\mu$, in the notation of the proof of Lemma 13. Then $X_{i-1} \to X_i$ is a closed immersion with open complement isomorphic to $\coprod_{l(\mu)=i} \mathbb{A}^i$. It is well-known that this implies our result. For the convenience of the reader, we review the standard argument. We have $M(\rho(\mathrm{Gr}_G)) = \mathrm{colim}_i M(X_i)$ (by Corollary 19), and $M(X_{-1}) = 0$, so it suffices to prove that $M(X_i) = M(X_{i-1}) \oplus \bigoplus_{l(\mu)=i} \mathbb{Z}[1/e]\{i\}[2i]$. Since each X_i is projective, we have $M(X_i) = M^c(X_i)$, where M^c denotes the motive with compact support [15, p. 9]. Thus we can use the Gysin triangle with compact support [8, Proposition 5.3.5]

$$M^c(X_{i-1}) \to M^c(X_i) \to M^c(X_i \setminus X_{i-1}) \to M^c(X_{i-1})[1].$$

Since $M^c(\mathbb{A}^i) = \mathbb{Z}[1/e](i)[2i]$, the boundary map vanishes for weight reasons (by induction, $M^c(X_{i-1})$ is a sum of Tate motives of weight $< i$), giving the desired splitting. □

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