(1) Let τ be a topology on $\mathcal{C}$. Denote by $L : Sp(\mathcal{P}(\mathcal{C})) \to Sp(\mathcal{S}\mathrm{hv}_\tau(\mathcal{C}))$ the left adjoint to the inclusion. Make sense of and prove the following statement: $L(\{X^n\}^n) \simeq \{L_\tau(X^n)\}^n$.

(2) Let $\mathcal{X}$ be an ∞-topos and let $Sp(\mathcal{X})_{<0}$ consist of spectra E such that $\Omega^\infty E = *$. Write $\mathcal{C}$ for the full subcategory of those $E \in Sp(\mathcal{X})$ which are left orthogonal to $Sp(\mathcal{X})_{<0}$ and also satisfy $\underline{\pi}_i E = 0$ for $i < 0$.

  (a) Given $E \in Sp(\mathcal{X})$, find $E' \in \mathcal{C}$ with a map $E' \to E$, surjective on $\underline{\pi}_i$ for all $i \geq 0$. [*Hint:* consider $E' = \Sigma^\infty \Omega^\infty E$.]

  (b) Let F be the fiber of $E' \to E$ and C_1 the cofiber of $F' \to F \to E'$. Show that $C_1 \in \mathcal{C}$ and $\underline{\pi}_0 C_1 \simeq \underline{\pi}_0 E$. Iterating this construction, find $C_\infty \in \mathcal{C}$ with a map $C_\infty \to E$ which induces an isomorphism on $\underline{\pi}_i$ for all $i \geq 0$.

  (c) Suppose now that $\mathcal{X}$ is hypercomplete. Show that $C_\infty \simeq \tau_{\geq 0} E$ and deduce the description of the t-structure in terms of homotopy sheaves.

(3) Let k be an infinite field, $w \in \mathbb{A}^n$ closed.

  (a) Let $x_1, \ldots, x_n \in \mathbb{A}^n$ closed points, with $x_i \neq w$ for all i. Show that for a general hyperplane H in $\mathbb{A}^n$, $w + H$ does not contain any x_i.

  (b) Let $Z \subset \mathbb{A}^n$ be closed of dimension > 0. Show that for a general hyperplane H, $\dim(Z \cap H + w) < \dim Z$.

(4) Let k be an infinite field and C a smooth curve over k. Let $z \in C$ be a closed point. Show that there exists an open neighborhood C' of z and a map $f : C' \to \mathbb{A}^1$ such that f is étale at z, and $f|_z : \{z\} \to \mathbb{A}^1$ is universally injective. (I.e. any base change of f_z along a map $T \to \mathbb{A}^1$ remains injective.) [*Hint:* you may wish to show that $X \to Y$ is universally injective if it is injective and all residue field extensions are purely inseparable.]

(5) Let $f : X \to Y$ be a morphism of schemes which is finite, unramified, and universally injective. Show that f is a closed immersion.

(6) Let $\varphi : U \to X$ be a separated étale morphism, $Z \subset U$ closed mapping isomorphically to its image in X. Show that there exists an open neighborhood V of Z in U such that $\varphi^{-1}(\varphi(Z)) \cap V = Z$. [In other words, $U \to X$ is an étale neighborhood of Z in a weak sense, and from it we extract $V \to X$, an étale neighborhood in a strong sense.]

(7) (a) Let G be a sheaf of groups on Sm_k. Show that G is strongly $\mathbb{A}^1$-invariant if and only if $K_{\mathrm{Nis}}(G, 1)$ $(:= L_{\mathrm{Nis}} K(G, 1))$ is $\mathbb{A}^1$-invariant.

  (b) Let A be a sheaf of abelian groups on Sm_k. Show that A is strictly $\mathbb{A}^1$-invariant if and only if $K_{\mathrm{Nis}}(A, n)$ is $\mathbb{A}^1$-invariant for all n.

(8) Let $Z \subset \mathbb{A}^N$ be a k-variety of dimension $\leq d$. Show that a general linear projection $\mathbb{A}^N \to \mathbb{A}^d$ induces a finite map $Z \to \mathbb{A}^d$, as follows. Given a linear map $\varphi : \mathbb{A}^N \to \mathbb{A}^d$, $\varphi_i(x_1, \ldots, x_N) = \sum_j \varphi_i^j x_j$, define homogeneous linear forms $\tilde{\varphi}_0 = X_0, \tilde{\varphi}_i = \sum_j \varphi_i^j X_j$. Set $V_\varphi = Z(\tilde{\varphi}_0, \ldots, \tilde{\varphi}_d) \subset \mathbb{P}^N$. We obtain a map $\tilde{\varphi} : \mathbb{P}^N \setminus V_\varphi \to \mathbb{P}^d$. Write $\overline{Z} \subset \mathbb{P}^N$ for the closure of Z.

  (a) Show that whenever $\overline{Z} \cap V_\varphi = \emptyset$, $\varphi|_Z : Z \to \mathbb{A}^d$ is finite. [*Hint*: show that $\varphi|_Z$ is projective and affine.]

  (b) Let G denote the Grassmannian scheme of $(N - 1 - d)$-dimensional linear subspaces of the hyperplane at infinity of $\mathbb{P}^N$. Let $T \subset \overline{Z} \times G$ be the set of pairs (x, V) with $x \in V$. Show that $\dim T < \dim G$. [*Hint:* Given a surjective morphism $X \to Y$ of varieties, such that all fibers have dimension $\leq n$, one has $\dim X \leq \dim Y + n$.] Deduce that $T \to G$ misses a dense open subset U. Show that $\{\varphi \mid V_\varphi \in U\}$ is the desired open.